Wave Transience in a Compressible Atmosphere. Part I: Transient Internal Wave, Mean-Flow Interaction\textsuperscript{1}

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ABSTRACT

Vertically propagating internal waves give rise to mean flow accelerations in an atmosphere due to the effects of wave transience resulting from compressibility and vertical group velocity feedback. Such accelerations appear to culminate in the spontaneous formation and descent of regions of strong mean wind shear. Both analytical and numerical solutions are obtained in an approximate quasi-linear model which describes this effect.

The numerical solutions display mean flow accelerations due to Kelvin waves in the equatorial stratosphere. Wave absorption alters the transience mechanism in some significant respects, particularly in causing the upper atmospheric mean flow acceleration to be very sensitive to the precise magnitude and distribution of the damping mechanisms.

Part II of this series discusses numerical simulations of transient equatorial waves in the quasi-biennial oscillation. These results are of sufficient qualitative interest to merit attention in this paper, and this is done with the help of a simpler, prototype standing-wave model (Plumb, 1977).

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i. Introduction

In the last two decades the theory of wave, mean-flow interaction has undergone considerable development. From the time of its inception the theory has sought to describe the interaction of waves and mean flows in terms of two processes. The first involves explicit departures from conservative motion, and is sometimes described under the more restrictive title "wave dissipation". The second involves departures from purely harmonic motion, such as wave growth and decay. This second category is comprehended under the title "wave transience". A third process, critical layer interaction, is sometimes considered separately but, in fact, should be included under both of the above categories.

The importance of wave transience and dissipation came to be appreciated first in a largely negative way, as a result of the pioneering work of Eliassen and Palm (1961) and Charney and Drazin (1961). In both papers a nonacceleration theorem was derived, showing how steady and conservative linear waves do not induce secondary changes in the mean flow. Since their work first appeared the nonacceleration theorem has been greatly generalized, culminating in the exact, finite-amplitude theorem of Andrews and McIntyre (1978a). The theorem in its present form not only embraces earlier theorems but also explains exactly, and in considerable detail, how nonconservative, non-steady waves alter mean flows.

The circumstances in which wave transience gives rise to mean flow changes are manifold. Instability is a prime example and is of extensive interest in meteorology. Problems involving forced waves are also common. In all these cases there is a nontrivial interaction of the waves with the mean flow in spite of the fact that the motion is completely conservative.

The purpose of this two-part study is to examine a type of wave transience problem that has been largely overlooked in the past, but is one that is beginning to be recognized as being of much interest in atmospheric dynamics. This problem involves the vertical propagation of forced disturbances in a compressible atmosphere. The motivation for this problem comes from the fact that when wave disturbances propagate upward in an atmosphere, there is an exponential growth of disturbance amplitude with height due to the basic-state density stratification. In many circumstances this growth contributes directly to the acceleration of the mean flow through the wave transience mechanism.

Part I discusses this problem in a rather general way through a study of the vertical propagation of a forced internal gravity wave. Analytical and numerical solutions are derived describing mean flow evolution within the context of an approximate model, the inherent assumptions of which are that the waves in question are linear and slowly varying.
In the latter part of this paper, and to a greater extent in Part II, application of these results is made to the theory of the stratospheric quasi-biennial oscillation.

The appendices contain results of peripheral interest to this study but, nevertheless, are of fundamental theoretical importance. Appendix A describes an analytic solution for a wave propagating towards its critical level. The resulting interaction, in contrast to existing critical layer theories (with their implicit assumptions of negligible forcing amplitude), is found to occur in the interior region, well below the critical level. There is in fact within our approximations no interaction at the critical level because the wave never reaches it.

Appendix B is written in a rather different spirit than that of the main study. There an attempt is made to describe the exact process by which a forced gravity wave transiently develops following switch-on of the forcing. Through the Laplace transform technique an exact solution is found which describes both the nature of the wavefront, and the various transients excited by the forcing. This solution takes the form of an integral of the zeroth order Bessel function.

In summary, the results of the theory developed in this study are of importance in a wide context, and exhibit some very fundamental aspects of wave transport. (A review text on this topic is currently in preparation by M. E. McIntyre.) Concepts such as radiation stress, wave action, pseudomomentum, and Kelvin's circulation are central to the theory and provide a physical basis for understanding the role of wave transience and dissipation in wave, mean flow interaction.

2. Single wave, mean-flow interaction

a. Qualitative discussion

The prototype problem involves a two-dimensional internal gravity wave excited by a small-amplitude, moving corrugation at the lower boundary, the details of which are sketched in Fig. 1. The fluid, which is an adiabatic, frictionless atmosphere of stable stratification $N^2 = \frac{gd \ln \theta_0}{dz} > 0$ and scale height $H$, is initially at rest, as is the corrugated lower boundary. At some initial time the corrugation begins moving with uniform velocity $c$ in the $x$ direction. As a result, an internal wave is excited in the region above the boundary. This wave, which we assume to be hydrostatic, for simplicity, has surfaces of constant phase tilted forward (consistent with the radiation condition). The height of the wavefront (i.e., the point which represents the primary increase in wave amplitude: in Appendix B this concept is given a precise definition) is given by the product of the vertical group velocity $k c^2 / N$ ($k$ is the wavenumber in the zonal direction) and the time $t$. Because of the decrease of density with height, the amplitude of the forced wave will grow with height, before approaching zero at the wavefront. To a first approximation, the growth of wave amplitude squared is equal to $\exp z / H$ in the lowest layers.

It is evident from Fig. 1 that the forced wave has the ability to induce a mean atmospheric motion in
the x direction, thereby altering its own propagation characteristics in the region. This result may seem surprising at first sight, since in a completely conservative system no mechanism of wave absorption exists (nor does any effective critical layer absorption exist in this problem, at least not in the initial state). However, examination of the figure reveals quite clearly that the corrugated boundary exerts a positive stress on the fluid above it. This stress results from a positive correlation between perturbation high pressure and negative lower boundary slope. Since no opposing stress is present above the wavefront, where the wave amplitude is zero (or, at least, very small), there results a net positive mean-flow acceleration in the interior. In other words,

$$\frac{dM}{dt} = B_0, \quad (1)$$

where the vertically integrated mean momentum $M$ is

$$M = \int_0^\infty \tilde{u}(z,t) \exp(-z/\ell) dz \quad (2)$$

and $B_0$ is the total radiation stress at the lower boundary. After switch-on of the forcing, this stress assumes a constant value independent of time, so that

$$M = B_0 t, \quad (3)$$

i.e., the total atmospheric mean momentum initially increases linearly with time. By implication, work must be done to keep the corrugation in motion; the force involved is like the familiar wave drag common to other problems of this kind.

Now if it be supposed that the formula (3) is valid for all time, there is in that case no limit to the possible eventual magnitude attained by the mean flow $\tilde{u}(z,t)$. In reality, however, there is a strong suggestion that an upper limit may be placed on the ultimate mean flow speed. The existence of such a limiting value follows from the theory of linear, slowly varying internal wave disturbances. In particular, when $\tilde{u}$ approaches the phase speed $c$, the vertical component of group velocity vanishes, i.e.,

$$W_g = \frac{k c^2}{N} \left(1 - \frac{\tilde{u}}{c}\right)^2 \to 0. \quad (4)$$

Within the context of the slowly varying approximation the upper limit on $\tilde{u}$ might also be inferred from the realization that, when the Doppler-shifted frequency vanishes locally at some level, the fluid above this level no longer feels a moving fluid corrugation below it. Consequently, there is no longer any vertical propagation of wave information or wave action into or through this level.

Now the existence of a limiting mean flow speed has serious implications, especially for a compressible atmosphere. This is because an atmosphere, for any finite mean flow speed, possesses only a finite total mean momentum, since it possesses only a finite total mass. This implies that

$$M \to \text{constant} \quad (5)$$

in time [contrary to (3)] which, in turn, necessitates that eventually

$$B_0 \to 0. \quad (6)$$

Thus at some finite time the wave fields must adjust so as to cause the lower boundary radiation stress to vanish. It is of interest to note that this result would not be expected in an infinitely deep, initially at rest Boussinesq fluid, since $M$ can be unbounded even though $\tilde{u}$ is bounded. (The initially at rest assumption is crucial to this statement; in Appendix A we explore a simple example of initial linear shear with a critical level. It is shown that the same reductio ad absurdum applies to that problem, together with its Boussinesq counterpart. What is of importance in both cases is the total effective mass involved.)

Stated differently, if the moving corrugation were to somehow continue to impart momentum to the atmosphere, this would require that eventually $\tilde{u}$ exceed $c$ somewhere. Since within the slowly varying, linear waves approximation such an occurrence will never be realized, it would follow that in fact the lower boundary must eventually cease to impart momentum to the atmosphere. Alternately, it might be inferred that if the lower boundary were to continue to impart this momentum, this would require an eventual breakdown in either or both the linear and slowly varying waves approximations.

Setting aside for the moment the possibility of a breakdown in these approximations, it is possible to infer not only the overall picture of $dM/dt$, but also the detailed profile of $d\tilde{u}/dt$, without extensive use of the mathematical equations, which are introduced in the next subsection. In particular, we expect to find a steady equilibrated solution initially in the lowest layers. This solution would grow with height slightly faster than $\exp(z/H)$, because of the initial radiation stress divergence on this part of the atmosphere. Above the equilibrated region lies the wavefront itself; rather than being a sudden jump from zero to $\exp(z/H)$, however, the wavefront would actually have a significant tilt, involving a gradual increase from the point $kc^2/N$ back to the equilibrated solution. The physical reason for this tilt lies in the fact that large-amplitude wave signals have a smaller vertical group velocity than small-amplitude signals. The reason for this behavior will be made clear below, but for now it is sufficient to say that it is required from Kelvin’s circulation theorem. The tilt is itself the reason why the equilibrated solution grows with height faster than $\exp(z/H)$, in view of the exact result (1).
Now for small amplitude forcings the tilt effect will not be very noticeable initially. Nevertheless, in an atmosphere there will always come a time when the tilt becomes so prominent that the atmosphere cannot continue to absorb momentum above the equilibrated region at the same rate it is being supplied at the lower boundary. In other words there would exist some height $z_0$ for which

$$\frac{d}{dt} \int_{z_0}^{\infty} \tilde{u}(z, t) \exp(-z/H) dz$$

cannot possibly equal $B_0$. It would then follow that the equilibrated solution is only temporarily valid.

Now the exact manner in which the equilibrated solution would break down is perhaps difficult to determine. Nevertheless, it seems plausible to suppose that a breakdown first occurs at $z_0$, the uppermost height attained by the equilibrated solution, followed by a breakdown in progressively lower layers. For levels below $z_0$ the final solution would then take the form of a descending shear layer, which eventually encounters the lower boundary.

### b. Analytical solutions

Whether or not the actual solution resembles this intuitive profile of $\tilde{u}$, depends on the accuracy of the linear and slowly varying approximations. It is nevertheless of some significance that the spontaneous formation of a descending shear layer is explicitly predicted by these approximations in a simple analytical model. We now proceed to develop the approximate governing equations of this model and solve the resulting single, quasi-linear equation by the method of characteristics. It will be observed that the solutions obtained in this way do not in themselves satisfy either of these approximations at large times. We lay aside for the moment, however, the question of precisely when these approximations break down, and examine the analytical solutions assuming the a priori validity of both approximations.

We begin by quoting an exact result

$$\frac{dC}{dt} = 0,$$  \hspace{1cm} (7a)

$$C = \int u \cdot ds,$$  \hspace{1cm} (7b)

where $C$ is Kelvin’s circulation taken over a material tube which is displaced by the internal wave (Fig. 2). (Note that by considering a periodic domain we need not close the material tube contour. Also, in the interests of preciseness, the integral is assumed to be normalized with respect to the total longitudinal distance covered. The material tube is isentropic in view of the initial condition.) The center-of-mass motion of a material tube is just the Lagrangian-mean velocity (Andrews and McIntyre, 1978a) (more precisely it is the Cartesian Lagrangian-mean velocity). In the absence of Coriolis effects and damping, $C$ is exactly conserved.

While the circulation theorem is an exact result, it will be employed in connection with the slowly varying and linear waves approximations. Now the first effect of these approximations is to suggest that $\tilde{u}^z$, the Lagrangian mean velocity, is just equal to the
Eulerian-mean velocity at lowest order, which is strictly zonal and of magnitude \( \tilde{u} \) (\( \tilde{w} \) is zero in this two-dimensional problem). There are two reasons for this result. First, there is no longitudinal Stokes drift \( \tilde{u}^L \) in our case (at lowest order); note that parcel motions are strictly rectilinear and not elliptical. Second, although it is entirely possible, in general, to have a nonzero \( \tilde{w}^L \) in two-dimensional problems of this sort (the Lagrangian-mean flow is not incompressible, especially when the waves are transient, as is true in the vicinity of the wavefront) the slowly varying approximation renders the magnitude of this term relatively small. Consequently, material tubes do not experience a significant change in their elevation, but are primarily carried along with the Eulerian-mean flow \( \tilde{u} \).

The linearization approximation suggests the usefulness of a Taylor series expansion of the circulation \( C \) in powers of the displacement field \( \xi \) [the latter is defined precisely in Andrews and McIntyre (1978a) but we employ the usual linearized fields \( \xi = (\xi_x, \xi_y, \xi_z) \) here]. This yields

\[
C = \tilde{u} + \xi_x \tilde{u}^t + (\tilde{u}^s + \xi_x \tilde{u}^t \tilde{u}_x) \quad (8)
\]

at lowest order. The three terms in parentheses are to be ignored under the present approximations, and although the final term could easily be included for nonhydrostatic disturbances, it is neglected here.

The term \( \xi_x \tilde{u}^t \) thus represents the leading wave contribution to the circulation, and is sometimes referred to as the leading contribution to the (minus) wave pseudomomentum, a conservative wave property in spatially symmetric mean flows (Andrews and McIntyre, 1978a,b). It may be written in the alternate approximate form

\[
-\xi_x \tilde{u}^t = \frac{\bar{u}_x^2}{\bar{u}^2} + \frac{\Delta x^2 / N^2}{2(c - \bar{u})} = A. \quad (9)
\]

The above equivalence results from 1) the linearized expression for the displacement field \( D_t \xi = u^t \) and 2) the equipartition of wave energy between kinetic and thermal components which is easily derived from the linearized, slowly varying wave fields. The symbol \( A \) is chosen insofar as the quantity in (9) is just the wave action apart from the factor \( k \) (Bretherton and Garrett, 1968) which in our case is irrelevant in view of our assumption of a single zonal wave-number.

The circulation theorem implies the remarkable result

\[
\tilde{u} = A, \quad (10)
\]

i.e., that the wave-induced mean flow is just equal to the wave action density, apart from an arbitrary constant which is zero in our case.

To this important result we append the usual mean flow equation

\[
\frac{\partial \tilde{u}}{\partial t} = \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0 B), \quad (11)
\]

where \( B \) is obviously the Reynolds stress \( \tilde{u}' \tilde{w}' \), but is also just the approximate radiation stress in our problem. This follows from the equivalence \( \tilde{u}' \tilde{w}' = \xi_x^t \tilde{w}^t = W_g A \), where \( W_g = k(c - \bar{u})^2/N \) is the vertical group velocity. (\( \phi \) is the geopotential and its vertical derivative is the hydrostatic temperature.)

The second quantity relates to our previous argument concerning the positive correlation between vertical displacement and the pressure gradient force, which is responsible for the work done on the fluid above the corrugated boundary. The third quantity states that the flux of wave action \( B \) is equal to the product of the vertical group velocity and the wave action density. This is a typical slowly varying result and follows from the simple relations between wave fields. It is thus apparent that

\[
B = \frac{kc^2}{N} A \left( 1 - \frac{A}{c} \right)^2 \quad (12)
\]

and that, consequently, the entire problem is described by a single equation for \( A \) (or \( \tilde{u} \)):

\[
\frac{\partial A}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \frac{k c^2 A}{N} \left( 1 - \frac{A}{c} \right)^2 = 0. \quad (13)
\]

By way of commentary on this result it may be useful to compare it to the traditional wave action equation

\[
\frac{\partial A}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \frac{k c^2 A}{N} \left( 1 - \frac{\tilde{u}}{c} \right)^2 = 0. \quad (14)
\]

It is important to realize that while (14) is a consistent result, in the sense that it is correct to leading order in wave amplitude, Eq. (13) is not a leading order result. It is, however, a consistent quasi-linear result, in the sense that the second-order mean flow (i.e., that due to the wave acceleration) is retained in the equation, while various wave-wave interactions are deliberately omitted. We wish to stress that this procedure, though somewhat arbitrary, is entirely consistent with the very philosophy of quasi-linear modeling employed so frequently in the literature, and for example in the original Holton-Lindzen (1972) paper dealing with the simulation of the quasi-biennial oscillation. The inherent usefulness of the quasi-linear assumption has been demonstrated many times in the literature; its usefulness in connection with the problem we wish to investigate here has not yet been established. Nevertheless, it will be worthwhile to pursue the quasi-linear assumption to its logical conclusion, while keeping in mind the deliberateness of the assumption itself.

In addition to being physically quasi-linear, the Eq. (13) is also mathematically quasi-linear, since although the solution \( A \) appears in a nonlinear way in the flux term, the derivatives of \( A \) are entirely
linear. As a result this equation is easily solved by the method of characteristics. Before doing so, however, it is convenient to rescale the variables appearing in (13). The obvious choice of scales is

\[ \frac{z}{H} \rightarrow z, \quad \frac{A}{c} \rightarrow A, \quad \frac{kc^2}{NH} \rightarrow t \rightarrow t. \]

For the remainder of this subsection the variables are nondimensional in accordance with this scaling. It is of some interest to note that the time scale is entirely independent of the magnitude of the lower boundary forcing.

There results the nondimensional equation

\[ \frac{\partial A}{\partial t} + \left( \frac{\partial}{\partial z} - 1 \right) A (1 - A)^2 = 0. \]  \hspace{1cm} (15)

Solutions are obtained by introducing a parameter \( s \) such that

\[ \frac{dA}{ds} = \frac{\partial A}{\partial z} \frac{dz}{ds} + \frac{\partial A}{\partial t} \frac{dt}{ds}. \]  \hspace{1cm} (16)

It follows that

\[ \frac{dA}{ds} = A(1 - A)^2, \]  \hspace{1cm} (17a)

\[ \frac{dz}{ds} = (1 - 3A)(1 - A), \]  \hspace{1cm} (17b)

\[ \frac{dt}{ds} = 1. \]  \hspace{1cm} (17c)

Characteristic curves are traced out by the equations

\[ t + c_1 = \ln \left( \frac{A}{1 - A} \right) + \frac{A}{1 - A}, \]  \hspace{1cm} (18a)

\[ z + c_2 = \ln [A(1 - A)^2]. \]  \hspace{1cm} (18b)

There is evidently just one, universal characteristic curve for this problem (apart from the two constants which are used to apply boundary and initial conditions). Fig. 3 displays this curve. An important feature of this characteristic is a maximum in \( z + c_2 \) occurring at \( A = \frac{1}{2} \) at which point

\[ t + c_1 = \ln \frac{1}{2} + \frac{1}{2}, \]  \hspace{1cm} (19a)

\[ z + c_2 = \ln \frac{1}{27}. \]  \hspace{1cm} (19b)

Particular solutions to (15) require the specification of initial and boundary conditions. From the theory of first-order equations it is well known that the space-time curve along which the solution is specified cannot be arbitrary if a unique solution is desired. Rather, initial and boundary conditions guaranteeing uniqueness depend in part on the precise nature of the characteristics.

It turns out that realistic conditions used in connection with (15) do not in general appear to produce unique solutions. Although it is possible to imagine very special cases in which unique solutions can be found, these cases are quite unrealistic, and do not correspond to any simple configurations of initial and boundary conditions, nor for that matter even the very simple conditions implied by Fig. 1. Thus a nonunique solution must be anticipated.
The conditions implied by Fig. 1 are

\[ A(z,0) = 0, \quad (20a) \]
\[ A(0,t) = A_B, \quad (20b) \]

where \( A_B \) is the nondimensional wave action density forcing amplitude. Figs. 4, 5 and 6 display the analytical solution for the cases \( A_B = 0.01, 0.033 \) and 0.10, respectively. The basic form of the solution in
all three cases involves 1) a wavefront located at \( z = t \), 2) an equilibrated solution beneath what we call the primary characteristic, and 3) a region bounded below by the primary characteristic and above by the line \( z = z_{\text{MAX}} \) in which the solution is double-valued.

The similarity between the three solutions is indeed remarkable, and is largely a consequence of the logarithmic dependence of \( z \) and \( t \) on \( A \). Another striking feature of the solutions is the upper region (above \( z_{\text{MAX}} \)) which essentially represents a continuing upper atmospheric response to the instantaneous switch-on. It is to be noted that there is complete “leakage” of the wave to \( z = \infty \), which is a consequence of both the absence of damping and the implicit neglect of any other wave attenuation/reflection because of the slowly varying approximation.

The presence of the nonunique region in all three cases is of course a manifestation of the physical constraint that the upper atmosphere cannot absorb mean momentum at the constant rate it is supplied at the lower boundary. As previously noted it is the quasi-linear effect of a tilted wavefront that is responsible for this effect. Had we employed the linear equation (14), without taking into account the second-order mean flow change, the wavefront would have assumed the form of a discontinuous step from zero to \( \exp(2/H) \). The corresponding solution for the mean zonal wind (and the wave action density) would have been in that case

\[
A_p \mathcal{H}(t - z) \exp(z/H),
\]

where \( \mathcal{H} \) is the Heaviside stepfunction. It is important to note that this solution is consistent with the requirements of the momentum budget, and thus might be a useful result in some circumstances; however, the linear result makes no attempt to account for mean flow induced changes in the wave which are central to our analysis.

The existence of this nonunique region, together with the physical circumstances underlying its presence, suggest the likelihood of a shock developing in the solution, beginning at \( z_{\text{MAX}} \). The development of such a shock, or discontinuity, places equation (15) in a class of partial differential equations which includes the more familiar Burgers’ equation, all of which permit the development of shocks under appropriate boundary conditions (Carrier and Pearson, 1976). The solutions to these equations can sometimes be made unique (and thus discontinuous) by imposing a shock condition of one sort or another. Such conditions are usually based on the physics of the particular problem. If we take the liberty of assuming the validity of the linear and slowly varying approximations after formation of the shock, the appropriate choice of shock condition would satisfy (1). One such condition, which itself turns out to be consistent with continuity of displacement field and total pressure across a vortex sheet (Acheson, 1976), is that \( B \) be continuous across the shock. With this condition the solution would then follow the primary characteristic down to the lower boundary. The solution above the primary characteristic would then become an extension of the radiating solution.
It is worthwhile to contrast these atmospheric solutions to those obtained in the Boussinesq limit. In that limit the entire physical space, as it were, collapses to a small square in the lower lefthand corner of Figs. 4–6. Characteristics are

$$\frac{dz}{dt} = -\frac{\partial A}{\partial t} = -\frac{\partial A}{\partial z} = (1 - A)(1 - 3A)$$  \hspace{1cm} (21)

and indicate a quasi-linear dispersion of the solution with height and time; no tendency for the spontaneous formation of shear zones is found in this purely transient, conservative, initially at rest Boussinesq case, given the conditions (20). (For an example of spontaneous shear zone formation in a Boussinesq case, the reader is referred to Appendix A, which considers the problem of initial linear mean zonal wind shear.)

c. Effect of damping

It is interesting to note that the formula (1) is valid whether or not wave absorption is present. The ultimate effect of damping, then, is not to alter $dM/dt$, but rather $\partial \vec{u}/\partial t$, the vertical profile of mean flow acceleration. Within our approximations, damping, if assumed to take the form of a weak (i.e., of order $\mu$, where $\mu$ is the measure of slowly varyingness) Rayleigh friction and Newtonian cooling of equal rate $\alpha$, alters the formula (13) to read

$$\frac{\partial A}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \frac{kc^2}{N^2} A \left( 1 - \frac{\bar{u}}{c} \right)^2 + 2\alpha A = 0.$$  \hspace{1cm} (22)

This result is based upon a straightforward extension of Kelvin's circulation theorem to the nonconservative case [it is also possible to derive this wave action equation from the linearized perturbation equations as in Andrews and McIntyre (1976a,b)] in which extra dissipative terms appear, viz.,

$$\frac{dC}{dt} + \oint \frac{dp}{\rho} + \oint \vec{X} \cdot d\vec{s} = 0.$$ \hspace{1cm} (23)

Now the friction $\vec{X}$ is analogous to $\vec{u}$ in (8) above (in fact, it is proportional to $\vec{u}$) so that clearly

$$\oint \vec{X} \cdot d\vec{s} = \xi_x X = -\alpha \frac{\bar{u}^2}{c - \bar{u}}.$$ \hspace{1cm} (24)

at lowest order since $\bar{X}$ is assumed equal to zero. The diabatic term may be evaluated by introducing an enthalpy $H$, and temperature $T$, together with entropy $S$, such that

$$\oint \frac{dp}{\rho} = \oint dH - \oint TdS = -\oint \vec{T} \vec{S} \cdot d\vec{s}.$$ \hspace{1cm} (25)

For conservative motion $S$ is conserved on each material tube, so that $\vec{S}$ is always perpendicular to $ds$ (and also assuming this is initially true, as we have from Fig. 1). But for nonconservative motion it is approximately true that

$$\oint \vec{T} \vec{S} \cdot d\vec{s} = -T' \frac{\partial S}{\partial x} = -\frac{TQ'}{c - \bar{u}}.$$ \hspace{1cm} (26)

where $Q'$ is the perturbation diabatic cooling, which if assumed proportional to $T'$ suggests that to leading order

$$\oint \frac{dp}{\rho} = -\alpha \frac{\bar{u}^2}{c - \bar{u}}.$$ \hspace{1cm} (27)

Hence

$$\oint \vec{X} \cdot d\vec{s} + \oint \frac{dp}{\rho} = -\alpha \frac{\bar{u}^2 + \bar{u}^2}{c - \bar{u}}.$$ \hspace{1cm} (28)

as in (22).

It will be observed that because of damping $\bar{u}$ is no longer equal to $A$ [cf. Eq. (22)]. As a result the system is now described by two coupled quasi-linear equations. Solutions to these equations have been obtained in a simple numerical model with finite differences in height and time. Numerical stability was achieved most easily by making these differences entirely forward, i.e.,

$$(dA)^n \to A^n - A^{n-1}.$$ \hspace{1cm} (29)

in height and time.

The values of model parameters have been chosen to be typical of the observed equatorial Kelvin wave (Wallace and Kousky, 1968). (The Kelvin wave, of course, is three dimensional in nature, but we here employ the prototype equatorial wave model of Holton and Lindzen, 1972, which bypasses discussion of the latitudinal structure by averaging across the equatorial waveguide, and ignoring latitudinal wind shear wherever it appears.) We take

$$k = (6.37 \times 10^6 \text{ m})^{-1},$$  \hspace{1cm} (30a)
$$c = 30 \text{ m s}^{-1},$$  \hspace{1cm} (30b)
$$A_B = 1 \text{ m s}^{-1},$$  \hspace{1cm} (30c)
$$N = 2 \times 10^{-3} \text{ s}^{-1},$$  \hspace{1cm} (30d)
$$H = 7 \text{ km},$$  \hspace{1cm} (30e)
$$\Delta t = 864 \text{ s},$$  \hspace{1cm} (30f)
$$\Delta z = 250 \text{ m}.$$ \hspace{1cm} (30g)

Following Holton and Lindzen (1972), a small eddy viscosity is also inserted into the mean flow equation, viz.,

$$\frac{\partial \bar{u}}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) B = \nu \frac{\partial^2 \bar{u}}{\partial z^2},$$ \hspace{1cm} (31)

where

$$\nu = 0.3 \text{ m}^2 \text{ s}^{-1}.$$ \hspace{1cm} (32)

That this, in fact, is a relatively small viscosity may be ascertained by comparing it with
\[
\frac{kc^2}{N} H = 49.4 \text{ m}^2 \text{ s}^{-1},
\]

the viscosity scale in the transience case.

Single wave experiments were performed over a large range of \( \alpha \), and the results are summarized here. Fig. 7a displays the numerical solution for the purely transient case. Qualitative agreement with the analytical solution is quite good, although the shear zone descent is a bit slower than that predicted by the above suggested shock condition. This is probably due to a combination of viscosity effects and truncation error. The primary effect of damping is a lowering in the altitude of maximum mean flow response, and a decrease in the rate of jet formation. Correspondingly there is less acceleration at upper levels with increasing \( \alpha \). As noted in a previous paper (Dunkerton, 1979) there is a critical value of \( \alpha \)

\[
\alpha_c = \frac{k c^2}{2 NH} = 5.05 \cdot 10^{-7} \text{ s}^{-1}
\]

such that (when \( \nu = 0 \))

\[
\lim_{z \to -\infty} \hat{u}(z,t) = \begin{cases} 
 c, & \alpha < \alpha_c \\
 0, & \alpha > \alpha_c.
\end{cases}
\]

The upper atmosphere response is therefore greatly affected by the precise magnitude and distribution of the damping mechanisms (Figs. 7b–7d).

d. Limitations of the present model

An important difference between transience-dominated and damping-dominated simulations concerns the maximum amplitude attained by the wave action density, \( A \). In the damping-dominated case there exists significant mean flow acceleration over an extended period of time even though the wave action amplitude is never large. Wave driving in the transience case, however, almost invariably involves relatively large wave action amplitudes at large times (nondimensional wave action amplitude approaching unity or greater). This implies the probable importance of nonlinear effects in the transience case. These effects could manifest themselves either in the form of wave, wave interactions in which higher harmonics are generated, or (and what seems more likely) simply in the turbulent breakdown of the primary wave due to local shear instabilities generated when the local Richardson number falls below \( \frac{1}{4} \).

The likelihood of such instability will be enhanced by the strong zonal wind shears which are observed to develop in both transience- and damping-dominated cases. The strongest shears seem to occur at, or very close to, the level of maximum wave amplitude. In a strongly damped situation, the local instability would arise in a largely zonally symmetric fashion, from the enhancement of the term \((\hat{u}_z)^2\).

On the other hand, in the purely transient case the instabilities would be much more localized in the wave, arising from the enhancement of the local shear \((\hat{u} + u')^2\).

The above remarks are not intended to rule out the possibility of significant wave-wave interactions. However, the slowly-varying approximation seems to mitigate against the occurrence of these interactions for internal waves. Nevertheless, the validity of this approximation in both cases is open to question (at least in the vicinity of the shear zones). Because the largest wave amplitudes are found in these regions, one cannot eliminate the possibility of the generation of higher harmonics.

Further investigation of this problem, especially its transience limit, is warranted. (Numerical experiments with a finite difference model are currently being performed by L. Coy at the University of Washington.) It is to be kept in mind, however, that all of the qualitative remarks in this section concerning the vertically-integrated mean flow acceleration \(dM/dt\), and also the local formula expressing conservation of Kelvin's circulation, are exact, and would apply in any model regardless of its degree of complexity.

3. Standing wave, mean-flow interaction

The theory of transient internal wave, mean-flow interaction described in the previous section is applicable to a wide assortment of atmospheric problems. The particular problem chosen for attention involved a hydrostatic internal wave of planetary scale, the prototype equatorial Kelvin wave, and the role of this wave in generating stratospheric mean flows was briefly discussed. This example appears to be directly relevant to the theory of the quasi-biennial oscillation (Holton and Lindzen, 1972; Plumb, 1977), as the following discussion will make clear.

In Part II, extensive results of numerical simulations of transient equatorial waves in the quasi-biennial oscillation will be presented. It is appropriate at this stage to describe some preliminary experiments involving transient, standing internal waves and their role in generating oscillatory mean flows of the type observed in the equatorial stratosphere. The following discussion is very much in the spirit of Plumb (1977) who examined the nontransient, damped wave problem.

a. Model equations

The single wave forcing of Fig. 1 is now replaced by a standing wave forcing. This wave is treated as the sum of two internal waves traveling in opposite directions, and mutual interactions between the two
Fig. 7(a–d). Numerical solutions for a transient, damped Kelvin wave, for various damping rates $\alpha$.
Note that a nondimensional unit of time is $\sim 11$ days for the model wave.
components are ignored. Because the problem is completely symmetric with respect to the origin, mean flows will not be generated unless there is some initial bias in \( \tilde{u} \). A suitable initial profile of mean zonal wind is chosen:

\[
\tilde{u}(z,0) = (z - 17)c/20, \quad 17 < z < 37 \text{ km}.
\] (36)

The entire problem is now described by three coupled quasi-linear equations:

\[
\frac{\partial A_+}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \frac{k c^2}{N} A_+ \left( 1 - \frac{\tilde{u}}{c} \right)^2 + 2\alpha A_+ = 0,
\] (37a)

\[
\frac{\partial A_-}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \frac{k c^2}{N} A_- \left( 1 + \frac{\tilde{u}}{c} \right)^2 + 2\alpha A_- = 0,
\] (37b)

\[
\frac{\partial \tilde{u}}{\partial t} + \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) (B_+ + B_-) = \nu \frac{\partial^2 \tilde{u}}{\partial z^2},
\] (37c)

where \( B_+ \) and \( B_- \) are the quantities appearing immediately to the right of the vertical derivatives in (37a) and (37b), respectively. The boundary conditions are

\[ A_+(0,t) = A_B, \] (38a)

\[ A_-(0,t) = -A_B, \] (38b)

\[ \tilde{u}(0,t) = 0. \] (38c)

Model parameters are identical to those listed above (30a–g).

**b. Numerical simulations**

To avoid overlap with Plumb’s (1977) experiments, attention has been directed exclusively at numerical integrations involving values of damping such that

\[ \alpha \leq \alpha_c. \] (39)

This criterion is intended to serve more or less as a guarantee that transience will be significant, as might be anticipated from the single wave experiments.

It should be noted that this criterion does not exclude the possible importance of wave transience when \( \alpha > \alpha_c \), for the following reason. Assuming that the ratio \( (A/c) \) is an appropriate measure of wave transience, one can estimate the maximum attained by this ratio in an arbitrary wind profile by assuming that the waves are steady; i.e., that

\[ B = B_0 \exp \left[ -\alpha \int (dz/W_\theta) \right] \] (40)

for damping independent of height. In the vicinity of the critical level,

\[ W_\theta(z) = W_\theta(z_c) + W_\theta'(z_c)(z - z_c) + \frac{1}{2} W_\theta''(z_c)(z - z_c)^2 \]

\[ = \frac{1}{2} W_\theta''(z_c)(z - z_c)^2 \] (41)

In this case, the maximum value of \( \alpha \) may be estimated as

\[ A \propto \frac{B_0 W_\theta''(z_c)}{\alpha^2}. \] (42)

It is apparent that in this Boussinesq case the maximum of \( A \) depends on the steepness of the barrier encountered by the wave as it approaches its critical level, and also upon the magnitudes of the forcing \( B_0 \) and damping \( \alpha \). R. A. Plumb (personal communication) has suggested that a number of Boussinesq cases exist for which the wave action density non-dimensionally approaches or exceeds unity in the late stages. On the basis of this result it must be concluded that the criterion involving the critical damping \( \alpha_c \) (35) merely serves to indicate the evolution of the mean flow at upper levels. This criterion does not serve to distinguish between transience-dominated and damping-dominated regimes. It does, however, guarantee the importance of transience in a compressible atmosphere.

It must also be said, however, that for the values of model parameters employed in this study, a damping-dominated regime begins to occur not long after the critical damping is exceeded. We have found, in both single and standing wave experiments, that for values of damping in excess of twice (roughly) \( \alpha_c \), the ratio \( (A/c) \) never attains a significant amplitude, at least not in our parameter range. This is no doubt coincidental to some extent, for such a result cannot be expected in the Boussinesq limit.

Fig. 8 displays an oscillatory solution for \( \tilde{u} \) with damping equal to \( 2 \times 10^{-5} \text{ s}^{-1} \) (independent of height). Examination of the solution reveals that \( (A/c) \) is approaching unity in the vicinity of the shear zone, and, consequently, wave transience is dominant. Because the damping is below the critical value, the mean flow oscillation is external in appearance in the sense that the magnitude of the oscillation is more or less independent of height over the range of the model. This provides a strong contrast to the Boussinesq oscillation (as for example was exemplified in the laboratory experiment of Gage and McEwan, 1978) in which the \( \tilde{u} \) maxima occur in the vicinity of the bottom boundary.

Fig. 9 shows a similar oscillatory solution for the case \( \alpha = 5 \times 10^{-8} \text{ s}^{-1} \) (independent of height). The solution, in fact, is almost identical to the previous one insofar as the mean zonal wind is concerned. There does appear to be a slight increase in both jet maxima and the local magnitude of the mean zonal shear in this case. A difference not seen in Figures is the decay time for the dominant wave at each shear zone which is four times as long in
lower $\alpha$ case. The oscillation as a whole, however, is remarkably independent of $\alpha$.

In the course of this investigation the question arose as to whether such oscillatory solutions would result in the purely transient case. Unfortunately, the standing wave model seems to require a small amount of damping for numerical stability, so that oscillatory solutions are always produced in the experiments. At this time it seems unlikely that the purely transient case would permit an oscillatory solution, since presumably some mechanism would be needed to remove the dominant wave in each shear zone.

c. Saturation hypothesis

In order to take account possible nonlinearity in transience-dominated cases, a saturation hypothesis has been introduced in a few of the experiments. The idea is to impose an upper limit on the magnitude of wave action density ($A_s$) beyond which the wave cannot grow because of nonlinear wave breaking. In the model, this was accomplished by resetting calculated values of $A$ back to $A_s$ whenever necessary.

Fig. 10 shows the nondimensional case of $A_s = \frac{1}{4}$ (other parameters identical to those of the previous
Fig. 10. As in Fig. 8 except with a saturation limit $A_s = \frac{1}{4}$.

figures). The damping is assumed equal to $2 \times 10^{-7}$ s$^{-1}$. The effect of this saturation has been a slight weakening in the strength of the oscillation, but overall there is much similarity to Fig. 8. Other values of $A_s$ produced similar results, and apparently indicates that this effect is able to mimic the usual transience/damping mechanism by creating an artificial gradient of wave-action flux resembling somewhat its usual shape.

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APPENDIX A

Analytic Solution with Initial Linear Shear

The analytic solutions presented in Section 2 assumed an initial condition of no motion. In this appendix we show that similar solutions exist for more complicated initial states. Of special interest is the initial condition

$$\tilde{u}(z,0) = \lambda z, \quad \lambda > 0,$$  \hspace{1cm} (A1)

$$A(z,0) = 0,$$ \hspace{1cm} (A2a)

$$A(0,t) = A_B,$$ \hspace{1cm} (A2b)

which for a single mode disturbance implies the existence of a critical level at $z = \lambda^{-1}$ (all quantities are nondimensional in accordance with the scaling of Section 2).

The circulation theorem again implies that

$$u = A + \lambda z \hspace{1cm} (A3)$$

for strictly conservative motion, subject to the linearization and slowly-varying approximations. Therefore,

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z} A(1 - A - \lambda z)^2 = A(1 - A - \lambda z)^2 \hspace{1cm} (A4)$$

which may be easily solved by the method of characteristics (the latter are determined numerically by integrating $z$ and $t$ with respect to $A$ for $0 < A_0 \leq A_B$, where $A_0$ is the starting value).

Fig. 11 displays the solution for the case $\lambda = 0.2$, $A_B = 0.01$. A critical level at $z = 5$ severely limits vertical propagation (Bretherton, 1966). Momentum input at the lower boundary must, however, be deposited somewhere, and this results in a mean flow acceleration which is largest at $z = 1.58$. This corresponds to the expected value of

$$z_{\text{max}} = \ln \left[ \frac{4 \lambda^3 (1 - \lambda z_{\text{max}})^3}{27 A_B (1 - A_B)^2} \right] \hspace{1cm} (A5)$$

with $A_B = 0.01$. At this level a shock first appears. Comparison to Fig. 4 indicates that shock formation has occurred well below, and slightly before, its appearance in the $u(z,0) = 0$ case. Assuming continuity of wave action flux at the shock, the asymptotic solution is

$$A = 1 - \lambda z, \hspace{1cm} (A6)$$

$$\tilde{u} = 1, \hspace{1cm} (A7)$$

for $0 < z \leq \lambda^{-1}$.

Lest it be thought that Boussinesq cases are uninteresting, it should be noted that the Boussinesq solution given these initial conditions is very similar
Fig. 11. Analytic solution for the case with initial linear shear, with $A_B = 0.01$.

to Fig. 11. In fact, the wavefront equation is identical in both cases, viz.,

$$\lambda' = \frac{1}{1 - \lambda z} - 1. \quad (A8)$$

In both cases the critical level acts as a barrier to vertical propagation, thus implying that the total mass affected by the wave is finite.

The atmospheric solution resembles in some respects the numerical integration performed several years ago by Jones and Houghton (1971). Those authors attempted to model the type of gravity wave critical layer interaction predicted by Booker and Bretherton (1967). However, when mean flow accelerations were allowed, a shelf was observed to form in the mean wind profile, which was located in the interior region well below the critical level (cf. their Fig. 3). While their result did not depend upon the slowly varying approximation, it is of interest to note that according to our theory a shock would form at the level $z_{\text{max}}$, which is equal to 31 km assuming the approximate values $A_B = 0.0001$ (the wave is assumed hydrostatic), $\lambda = 0.12$, and $H = 6.6$ km. Jones and Houghton observed shelf formation at 34 km. While it is difficult to determine when this shock formed, we note that a nondimensional unit of time in model is $-11$ min, and with $A_B = 0.0001$ shock development would presumably occur between 10 and 20 time units or about 2–4 h, in approximate agreement with their result.

It is interesting to note that the Jones-Houghton shelf did not descend in time (although the numerical integration was terminated not long after its development). Whether this discrepancy is due to an inability of their model to correctly resolve this discontinuity, or is due to some physical process, remains to be determined.

The theoretical result displayed in the figure reminds us, incidentally, of a crucial assumption of the nonlinear critical layer theories; namely, that the forced wave be infinitesimal. In our result the limit $A_B \to 0$ would imply that the locus of interaction moves arbitrarily close to the critical surface, thus implying the possibility of a genuine critical layer interaction, a la Booker and Bretherton (1967). Presumably some sort of uncertainty principle (i.e., a breakdown in the slowly varying concept of a rigid wavefront) would allow such an interaction to take place.

At the time of this writing, the numerical experiments of L. Coy have demonstrated a remarkable degree of correspondence to the type of mean flow evolution predicted by our Fig. 11, insofar as there seems to be a considerable amount of interaction of the wave with the mean flow in the interior region, well below the critical level.

APPENDIX B

Exact Solution When $\tilde{u} = 0$

Because the results of the analytic theory of Section 2 depend in a large measure on the precise definition of such slowly varying concepts as for example that of a wavefront, it is worthwhile to investigate, at a deeper level than that afforded by the slowly varying approximation, exactly what motion results
from the instantaneous switch-on of the corrugated boundary of Fig. 1. To do this we shall neglect the induced mean flow altogether, and employ a Laplace transform technique to arrive at an exact solution for the hydrostatic temperature field valid for all time.

The simplest differential equation governing hydrostatic internal waves is the fourth-order equation

$$\phi'''' + N^2 \phi' = 0$$  \hspace{1cm} (B1)

valid for Boussinesq disturbances, where $\phi'$ is the geopotential in log-pressure coordinates. This equation is also valid for locally Boussinesq disturbances, that is, disturbances whose vertical wavelength is less than the density scale height of the atmosphere, provided that the exponential growth factor is removed from the equation, as is usually done via the substitution

$$\phi' = \text{Re} \phi \exp \left( i k x + \frac{z}{2 H} \right).$$  \hspace{1cm} (B2)

A Laplace transform

$$\hat{\phi} = \int_0^\infty \phi(t) \exp(-st) \, dt, \quad \text{Re} s > 0,$$  \hspace{1cm} (B3)

results in the second-order equation

$$s^2 \phi'' + \frac{k^2 N^2}{s^2} \phi = 0,$$  \hspace{1cm} (B4)

the solution of which is

$$\phi = A \exp \left( \frac{kNz}{s} \right) + B \exp \left( -\frac{kNz}{s} \right).$$  \hspace{1cm} (B5)

The vanishing of $\phi$ at $z = \infty$ demands that $A = 0$ since $\text{Re} s > 0$; at the lower boundary,

$$\phi(0,t) = \phi_B \exp(-i\omega t), \quad t > 0,$$  \hspace{1cm} (B6)

and hence

$$\hat{\phi}(0) = \frac{\phi_B}{s+i\omega} = B,$$  \hspace{1cm} (B7)

so that

$$\hat{\phi}(z) = \phi_B \frac{\exp \left( -\frac{kNz}{s} \right)}{s + i\omega}.$$  \hspace{1cm} (B8)

We note that by taking the vertical derivative of (B8) there results a well-known transform

$$\hat{\phi}_z = \frac{kN\phi_B}{s + i\omega} \exp \left( -\frac{kNz}{s} \right).$$  \hspace{1cm} (B9a)

$$= -\frac{kN\phi_B}{s + i\omega} L[J_0(2\sqrt{kNz})].$$  \hspace{1cm} (B9b)

The convolution theorem thus implies that for all time,

$$\phi_z = -\frac{kN}{\omega} \phi_B \exp(-i\omega t) I(\omega t, mz),$$  \hspace{1cm} (B10)

where

$$I(T,Z) = \int_0^T \exp(iT')J_0(2\sqrt{ZT'})dT'.$$  \hspace{1cm} (B11)

This integral describes the vertical variation, together with the transient adjustment. [Solutions to (B1) are quoted by Lighthill (1969) in a different physical context.] The behavior of this integral may be anticipated from (B11); the exp and $J_0$ most of the time are oscillating at different frequencies and hence do not contribute systematically to the integral, except when $Z$ approaches $T$. This point, as expected, is the point for which $(z/t) = \omega^2/Nk$; i.e., denotes the passage of the wavefront. Fig. 12 shows an example in the complex plane for the case $Z = 20$. The asymptotic behavior is

$$I(\infty,Z) = i \exp(-iZ),$$  \hspace{1cm} (B12)

as is required to form the steady, simple plane wave. Interestingly, we also find that

$$|I(Z,Z)| = 1/2,$$  \hspace{1cm} (B13)

which is an enlightening commentary on the slowly varying wavefront in which the solution jumps suddenly from zero to one.

That (B11) is indeed a solution to (B1) is not difficult to prove. Whenever $w = J_0(\eta)$, where $\eta = 2(ZT)^{1/2}$ it follows that $w_{zt} = -w$, and hence that $w_{zt} = w$.

We note in conclusion an interesting paradox posed by (B11). In an atmosphere, the initial wiggles of the Bessel function will grow with height, in spite of their transient decay [cf. Eq. (B2)]. Hence, if this result were correct, we would have the frightening conclusion that infinite wave amplitudes, and hence infinite mean flow accelerations, would occur instantaneously at infinite heights!

This paradox is resolved, however, by noting that the locally Boussinesq assumption, while valid for the forced plane wave, is not valid for the long-
vertical-wavelength initial oscillations. The equation governing hydrostatic atmospheric disturbances is more exactly

$$\phi_{zztt} - \phi_{tt}/4H^2 - k^2N^2\phi = 0 \quad (B14)$$

the Laplace transform solution of which is now

$$\hat{\phi} = \hat{\phi}(0) \exp \left[ - \left( \frac{1}{4H^2} + \frac{k^2N^2}{s^2} \right)^{1/2} \right] z. \quad (B15)$$

Evidently, the initial oscillations are evanescent, decaying like \(\exp(-z/2H)\). This implies the more reasonable result that only finite accelerations occur initially, and that these accelerations appear at finite heights, in the vicinity of the wavefront.

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