Some Eulerian and Lagrangian Diagnostics for a Model Stratospheric Warming

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ABSTRACT

Some new diagnostics are presented for a wavenumber-2 sudden warming, simulated by a version of Holton's semi-spectral, primitive-equation model. First, Eliassen-Palm cross sections exhibiting the Eliassen-Palm (EP) planetary-wave flux together with contours of the corresponding flux divergence, are presented for selected days of the simulation. Second, a description of zonal-mean-flow evolution in the model, simpler than the conventional Eulerian-mean description and qualitatively like Lagrangian-mean descriptions in some respects, is constructed from the transformed Eulerian-mean equations presented by Andrews and McIntyre (1976). In this description the mean warming is brought about by a thermally direct "residual meridional circulation" arising as an essentially adiabatic response to a wave-induced torque about the earth's axis. The torque itself is equal to the divergence of the EP wave flux and approximately proportional to the northward flux of quasi-geostrophic potential vorticity. Third, some true Lagrangian means and related diagnostics are presented and discussed.

The EP cross sections strikingly display the effect of the mid-stratospheric zero-wind line which invades middle latitudes from the tropics during the first stage of substantial evolution of the mean state. This zero-wind line develops into a partial reflector of planetary waves, splitting the EP wave flux into two branches and deflecting one of them equatorward and the other to high polar altitudes. The consequent focusing of waves into a smaller horizontal area in the polar cap and into latitudes with lower densities helps bring about the reversal of the polar westerlies in the second stage of mean evolution. Focusing of planetary waves into the high-altitude polar cap should be similarly important for real warmings, but there is no evidence that subtropical zero-wind lines play any important role. Possible mechanisms leading to focusing and hence to warmings in the real atmosphere are discussed.

To picture the model warming in Lagrangian terms, we first compare the shape of an isentropic surface near the level of maximum warming with the computed behavior of sets of air parcels. The isentropic surface is approximately a material surface over the short times concerned. As the warming develops the surface dips down over the pole and rises at the equator (and in the real atmosphere this leads to widespread cooling in the summer stratosphere as has often been observed). Thermally direct motion similarly appears in the residual, generalized Lagrangian-mean, and modified Lagrangian-mean meridional circulations near the level of maximum warming, as might have been expected from the theoretical results of Matsuno and Nakamura (1979). The divergence effect, or non-solenoidality of Lagrangian-mean motion, neglected in their study, is strong here because of the large north-south dispersion of air parcels accompanying the highly transient wave activity. Implications for modeling tracer transport are noted.

1. Introduction

Among numerical experiments of the sort pioneered by Matsuno (1971) as hypothesis-testing models of stratospheric warmings, there is one example for which an unusual amount of Eulerian and Lagrangian diagnostic information has become available (Hsu, 1980, hereafter referred to as I). The numerical model used is an adaptation of the truncated, semi-spectral, primitive-equation model of Holton (1976), and the reader is referred to Holton's paper, together with I, for details of the model. The availability of the diagnostic information invites a deeper probe into the model dynamics, with the aim of improving our understanding not only of the model itself, but also of the differences between such models and the real atmosphere, as discussed, for example, by Matsuno (1971), Quiroz et al. (1975), Schoeberl (1978), and in the following paper by Palmer (1981a). These differences seem most striking when zonal wavenumber 2 dominates (the example studied in I); and so we concentrate on this example in the present paper also.

The model diagnostics turn out to be interesting in several ways. First, we discuss in Section 3 the Eliassen-Palm cross sections for the model [for detailed motivation and a review of the underlying...
theory see Edmon et al. (1980)); the necessary formulas are recapitulated in Section 2. EP cross sections conveniently display the principal eddy fluxes of heat, momentum and quasi-geostrophic potential vorticity on one diagram. Moreover, the cross sections directly visualize the vertical and meridional propagation of planetary waves in a way that is directly relevant to their effects on the mean state. No assumptions about wave steadiness, etc., are involved.

The patterns of wave propagation exhibited by the EP cross sections suggest why this kind of model evolves toward the simulated wave-2 warming event in two distinct stages of which the first seems to have no precise counterpart in the real atmosphere, as originally remarked on by Matsuno (1971) and confirmed by the work of Holton (1976) and Schreiber and Strobel (1980a, Figs. 4, 5). Reflection from a nonlinear critical layer appears to play a significant role, an interpretation which is supported by comparing the Lagrangian and quasi-Lagrangian diagnostics of I with the theory of nonlinear critical-layer reflection. This raises several interesting questions concerning the numerical simulation of nonlinear critical layers as well as their possible role in the real atmosphere (cf. Béland, 1978; Tung and Lindzen, 1979a,b; Tung, 1979; Appendix B below).

In Sections 4 and 5 we use the EP diagnostics to follow the evolution of the mean flow through into the second stage of the model warming, which includes the reversal of the polar westerlies and which appears to have more in common with observed warmings. To good approximation, the divergence of the EP wave flux is proportional to the northward eddy flux of quasi-geostrophic potential vorticity and can be regarded as a torque on the zonal mean state representing the total mean effect of the waves. The transformed Eulerian-mean equations presented by Andrews and McIntyre (1976, 1978a) provide the clearest expression of this useful fact, and form the basis for our discussion. For relevant background, see also Charney and Drazin (1961), Dickinson (1969), Green (1970), Holton (1974, 5), Boyd (1976), Rhines (1977), Dunkerton (1978), Wallace (1978), Rhines and Holland (1979), Matsuno (1980), and the reviews in Holton (1980) and in Edmon et al. (1980).

As a result of adopting the viewpoint suggested by the transformed equations we obtain an especially clear picture of the warming dynamics. In the second stage of the model warming, for instance, the decelerating torque per unit mass is large throughout a deep layer in the middle atmosphere, spanning several scale heights. As already suggested by the work of Geisler (1974), Holton (1976) and Holton and Dunkerton (1978), the situation is then at an opposite extreme to that described by critical-layer theory for the zero-wind line over the polar cap. Critical-layer theory, as used for instance by Matsuno and Nakamura (1979), implies only a shallow layer of wave-induced torque confined to the neighborhood of that zero-wind line. Neither a linear nor a reflecting, nonlinear critical layer has time to develop over the polar cap until well after the polar westerlies have reversed.

The conceptual simplifications introduced by the transformed Eulerian-mean equations are, on the other hand, broadly similar to those achieved by Matsuno and Nakamura (1979, hereafter MN) with a Lagrangian-mean formulation, as is discussed further in Section 4. Like MN's Lagrangian-mean meridional circulation, the "residual circulation" entering the transformed equations arises as an essentially adiabatic response to the wave-induced torque. Its descending branch near the pole is responsible for the large rise in mean temperature there. The transformed equations have the advantage of not depending on special assumptions concerning wave amplitude, transience or dissipation such as those made by MN. Their usefulness has been further illustrated in concurrent modeling studies by Holton and Wehrbein (1980) and Hsu (1981). The equations can be expected to be useful for the real atmosphere also, an expectation already borne out by the work of Palmer (1981a) on the observed wave-2 warming of February 1979. Their approximate relationship with the equations of Lagrangian-mean theory has been discussed in another context by Dunkerton (1978).

Lagrangian information is of fundamental interest in its own right, and among other things permits a direct check on the extent to which MN's ideas, taken literally, apply to the simulated warming. In the second half of this paper, Sections 6–9, we present some results on Lagrangian and quasi-Lagrangian diagnostics. Section 6 begins by discussing the simplest quasi-Lagrangian aspect of the warming, namely, the motion of a typical isentropic surface near the level where warming is at a maximum. As already noted in I, and well illustrated by Fig. 9, this approximates the motion of a material surface. The warming is again seen to be an essentially adiabatic process, as is the complementary tropical cooling familiar from satellite observations (Fritz and Soules, 1970, 1972; Barnett, 1975; Quiroz, 1979a). We note in passing (to answer a question raised by Quiroz) the reasons for the observed global extent of the complementary cooling in the real summer stratosphere.

Calculations of generalized Lagrangian-mean motions near the level of maximum warming are presented in Sections 7 and 8. The calculated mean motions resemble those of MN's small-amplitude model in important respects, particularly descent in the polar cap (opposite to the Eulerian-mean circulation, but similar to the residual circulation) and ascent in the tropics (where all three circula-
tions agree qualitatively since wave activity is weak there). The pattern is similar in essence to that deduced in the pioneering observational study by Mahlman (1969), in which means were taken along the jet stream as well as zonally.

In addition we find (i) that the generalized Lagrangian means are much less smoothly behaved than in MN's small-amplitude case, and also in comparison to the residual circulation in the present model, and (ii) that the "divergence effect" is important, i.e., non-solenoidality of the Lagrangian-mean flow due to air-parcel dispersion (McIntyre, 1973, 1980a; Andrews and McIntyre, 1978b, Section 9). The divergence effect modifies the Lagrangian-mean northward flow in the deep layer that feels the wave-induced torque (cf. MN), in such a way that the northward flow tends to be reduced in high latitudes and enhanced in lower latitudes in comparison to what might have been expected from a superficial examination of MN's results. In their model the northward Lagrangian-mean flow, being concentrated within a horizontal critical layer, is strong compared with the divergent component.

One of the practical difficulties in using generalized Lagrangian means is that they can be defined only relative to a suitable initialization procedure, actual or hypothetical (Andrews and McIntyre, 1978b). For simulations like the present one there is a natural choice of initialization, based on the undisturbed state from which the simulation itself is initialized. This is not true of the real atmosphere, nor of long-term numerical simulations. The only way so far suggested of overcoming the resulting ambiguity is to use the "modified Lagrangian mean" discussed by Dunkerton (1980) and McIntyre (1980a,b). An equivalent quantity was introduced earlier by Obukhov (1964, 1971) in a different context. In Section 9 we present some computations of modified Lagrangian means and note some difficulties which they, too, pose in practice.

Section 10 concludes by returning to the differences between the behavior of the model and the real atmosphere. We discuss some observational evidence which appears to throw light on the differences, and which together with the present diagnostics helps to identify the questions deserving closest attention in future work.

2. EP cross sections: Basic definitions

As in 1 we use

\[ z = -H \ln(p/p_s) \quad (2.1) \]

as vertical coordinate, \( p \) being pressure and \( p_s \) a standard constant pressure, here taken as 1000 mb;

\[ H = R T_s g \]

is a standard constant scale height, here taken as 7 km, corresponding to \( T_s = 239 \) K. \( R \) is the gas constant, and \( g \) the gravitational acceleration. The standard density corresponding to \( p_s \) and \( T_s \) is

\[ \rho_s = p_s/R T_s = p_s/g H. \]

We may call \( z \) "pressure altitude" or simply "altitude." With this coordinate, the quasi-geostrophic approximation to the Eliassen-Palm wave flux (Edmon et al., 1980, and refs.) may be defined as

\[ \mathbf{F} = \{ F_{(\theta)}, F_{(z)} \}, \]

where

\[ F_{(\theta)} = -\rho_0 a \cos \varphi (\bar{v}' \theta'), \quad (2.2a) \]
\[ F_{(z)} = \rho_0 a \cos \varphi (\bar{v}' \bar{\theta})/\bar{\theta}_z, \quad (2.2b) \]

\[ \rho_0 = \rho_0(z) = \rho_0 e^{-z/H}. \]

Here subscripts such as \( z \) in \( \theta \) denote partial differentiation, \( \varphi \) is latitude, overbars and primes denote zonal means and departures therefrom, \( \theta \) potential temperature, \( a \) the radius of the earth, \( f = 2\Omega \sin \varphi \) the Coriolis parameter, and \( u \) and \( v \) the zonal and meridional velocity components. The convention is slightly different from that of Edmon et al., insofar as the factor

\[ p = p_s e^{-z/H} \]

arising from the use of logarithmic pressure coordinates (Edmon et al., 1980, Section 7) is incorporated \textit{ab initio} into the definitions (2.2) along with a constant factor \( \rho_0/\rho_s \) which is included for dimensional convenience. If the actual density scale height \( RT/g \) were equal to \( H \) everywhere (implying an isothermal atmosphere), then \( z \) would be equal to geometric height, measured from the level at which \( p = p_s \), and \( \rho_0(z) \) would equal the density at height \( z \). Our model atmosphere is not exactly isothermal, but its \textit{global average} on isobaric surfaces is specified as isothermal with \( T = T_s = 239 \) K, corresponding to a scale height of 7 km and a buoyancy (Brunt-Väisälä) frequency of \( 2.0 \times 10^{-2} \) s\(^{-1}\).

As was pointed out in Edmon et al., \( \mathbf{F} \) is parallel to the meridional projection of the planetary-wave group velocity, in cases where the notion of group velocity is well defined. The same is true of the eddy energy flux or eddy flux of geopotential, which for a steady, conservative wave of constant zonal phase speed \( c \) equals \(( \bar{u} - c ) \mathbf{F} \) (Eliassen and Palm, 1961; Boyd, 1976). But \( \mathbf{F} \) has the fundamental advantage that its divergence

\[ \nabla \cdot \mathbf{F} = \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left( F_{(\theta)} \cos \varphi \right) + \frac{\partial}{\partial z} (F_{(z)}) \quad (2.3) \]

is zero under "nonacceleration conditions" [steady, conservative waves on a steady mean flow (see the discussion, and references to past contributions, in Edmon et al.)]. This is the celebrated Eliassen-Palm theorem. Thus, an EP cross section, in which \( \mathbf{F} \)
is represented by arrows and \( \nabla \cdot \mathbf{F} \) by contours, displays information not only about the net direction of wave propagation, but also about the precise locations where nonacceleration conditions are being violated. In the case of MN's model, for example (and other models based on linear critical-layer theory), the contours of \( \nabla \cdot \mathbf{F} \) would all be crowded into a thin region surrounding the zero-wind line upon which the planetary waves are incident.

Similar results hold without making the quasi-geostrophic approximations [but with more restrictive assumptions on wave amplitude (again, see Edmon et al.)]. The ageostrophic forms of (2.2a, b) are given in Appendix A as Eqs. (A1a, b). In the course of the present investigation, EP cross sections were constructed from (A1) as well as from (2.2). The differences were quantitatively noticeable, but for the most part qualitatively insignificant. All the results to be presented here were calculated from the simpler forms (2.2).

The graphical conventions follow, in essence, those of Edmon et al., with appropriate modifications for the z coordinate defined by (2.1). The "volume" element for integrating (2.3) over a zonally symmetric portion of the atmosphere is

\[
dV = 2\pi a^2 \cos \varphi d\varphi dz. \tag{2.4}
\]

From (2.3) and (2.4),

\[
\int \nabla \cdot \mathbf{F} dV = \int \Delta d\varphi dz, \tag{2.5}
\]

where

\[
\Delta = \frac{\partial}{\partial \varphi} \left[ 2\pi a \cos \varphi F_{(\varphi)} \right] + \frac{\partial}{\partial z} \left[ 2\pi a^2 \cos \varphi F_{(z)} \right]. \tag{2.6}
\]

Here \( \Delta \) is the natural form of the divergence of \( \mathbf{F} \) for contouring in the \( (\varphi, z) \) plane, since by (2.5) the volume between that plane and a surface at distance \( \Delta(\varphi, z) \) from it equals \( \int \nabla \cdot \mathbf{F} dV \). The arrows will be drawn with horizontal and vertical components proportional to the quantities within square brackets in (2.6), viz.,

\[
\left\{ F_{(\varphi)}, F_{(z)} \right\} = 2\pi a^2 \cos \varphi \left\{ a^{-1} F_{(\varphi)}, F_{(z)} \right\}, \tag{2.7}
\]

expressed in terms of the scale units for \( \varphi \) and \( z \) in the diagram. For example, to calculate the horizontal and vertical arrow components as measured on the diagram, when \( F_{(\varphi)} \) and \( F_{(z)} \) are evaluated in SI units, their respective numerical values are multiplied by factors proportional to the distances occupied on the diagram by one radian or 57° of latitude, and one meter of pressure-altitude. This determines the directions and relative magnitudes of the arrows uniquely. Eq. (2.6) then implies that the pattern of arrows will look nondivergent if and only if \( \nabla \cdot \mathbf{F} \) is zero.

Before proceeding to the model diagnostics in the next section we note an alternative form of (2.2) which arises if \( u \) and \( v \) are themselves evaluated geostrophically, in terms of the geopotential \( \Phi \). Since

\[
\theta = T \exp(\kappa z/H) = T(\hat{\theta}/\hat{T}) = \Phi_x(H\hat{\theta}/R\hat{T}),
\]

from the hydrostatic relation, where \( \kappa = (\gamma - 1)/\gamma = 2/3 \) and \( \gamma \) is the ratio of specific heats, it follows that

\[
\theta'/\hat{\theta}_x = \Phi_{x'}/N^2, \tag{2.8}
\]

where

\[
N^2(y, z) = \frac{R \hat{T}}{H \hat{\theta}_x} = \frac{R}{H} \left( \hat{T}_x + \frac{\kappa \hat{T}}{H} \right). \tag{2.9}
\]

Substituting the geostrophic relations \( f u' = -\Phi_v' \) and \( f v' = \Phi_x' \) into (2.2) yields

\[
F = \rho_0 a \cos \varphi \left\{ \frac{\Phi_x' \Phi_v'}{f^2}, \frac{\Phi_x' \Phi_x'}{N^2} \right\}, \tag{2.10}
\]

where \( x \) is longitude (rad) multiplied by \( a \cos \varphi \), and \( y \) latitude times \( a \). Relation (2.9) implies that \( N \) is equal to the buoyancy frequency of the zonally averaged state wherever \( R \hat{T}/g = H \); the notation is chosen to suggest the relationship with that of Holton (1975, 1976).

Eq. (2.10) is convenient for relating \( F \) to the phase \( \phi \) of a zonal harmonic component of the planetary wave field. If \( \Phi' = |\Phi(y, z)| \cos(kx + \phi(y, z)) \), then we have

\[
\Phi_{x'} \Phi_{x'} = \frac{1}{2} k |\Phi|^2 \phi_x, \tag{2.11}
\]

together with a similar result with \( y \) in place of \( z \). There is no primae facie reason why the direction of \( \mathbf{F} \), which is related to the group velocity, should bear any simple relation to the phase contours \( \phi(y, z) = \) constant. However, if the EP cross sections are plotted with the foregoing conventions on a diagram in which the vertical scale is stretched with respect to the horizontal scale by a factor equal to the value of \( N/f \) at some height and latitude, then the wave propagation properties will appear to be locally isotropic in \( y \) and \( z \). At that height and latitude the arrows representing \( \mathbf{F} \) will appear to be at right angles to the phase lines on the diagram, by (2.11) et seq., but will be more nearly vertical in poleward latitudes for given \( N \), and more nearly horizontal in equatorward latitudes. Note that these formulas do not apply to equatorially trapped planetary waves, for which ageostrophic terms such as \( \omega' u' \) in Eq. (A1b) play an essential role.

We have not yet made the approximations in the thermodynamic equation introduced by Holton (1975, p. 32; 1976, p. 1640), but the model itself does make those approximations, which have the effect of replacing \( N^2 \) by its global average value \( 4 \times 10^{-4} \text{ s}^{-2} \).
In evaluating (2.2) and thence (2.6) and (2.7), we use the values of $u'$, $v'$ and $\Phi'$ output by the model, and evaluate $\theta'/\theta_z$ from (2.8) with $4 \times 10^{-4}$ $s^{-2}$ substituted for $N^2$.

3. EP cross sections: Results and discussion

EP cross sections were computed for each day of the model simulation described in I. Three examples, for days 11, 15 and 22, are presented in Fig. 1. They reveal that, early in the first stage of substantial mean-flow evolution, between days 10 and 20, the EP wave flux splits into two branches with one branch going equatorward and the other more nearly vertically. This striking phenomenon is well illustrated by Figs. 1b and 1c. Fig. 1a, and its predecessors (not shown) display no obvious splitting; in fact the EP cross sections up to day 11 merely show

![Fig. 1. Eliassen-Palm cross sections for the lowest 50 km of the model: (a) day 11, (b) day 15 and (c) day 22 of the model's evolution. The heavy curve in (b) shows part of the zero-wind line for day 15. The contours represent the quantity $\Delta$ defined by (2.6); the numerical values marked on the contours are to be multiplied by $2\pi a^2 \rho_0 \times 10^{-7}$ $m$ $s^{-2}$. The contour interval is 2 units. "Height" means pressure-altitude $z$ in kilometers, as defined by (2.1). The arrow scales are the same in all three cross sections; and such that the distance occupied by 10° of latitude represents a value $2\pi a^2 \rho_0 \times 5.559$ $m^2 s^{-2}$ of $F_{\theta\theta}$, and that occupied by 10 km of pressure-altitude $z$ represents a value $2\pi a^2 \rho_0 \times 0.05$ $m^2 s^{-2}$ of $F_{\theta z}$; where $F_{\theta\theta}$ and $F_{\theta z}$ are defined in (2.7).]
linear, transient wave development, beginning with
the initial upward propagation from the lower
boundary followed by refraction and diffraction
toward the tropics as originally found by Matsuno
(1970). The subsequent split into equatorward and
upward branches seen in Fig. 1b is very persistent
once established, and lasts throughout the second
and final stage of the warming (days 20–26). It is
prominent in Fig. 1c, for example, which is the EP
cross section for day 22, just before the final appear-
ance of easterlies throughout most of the polar
middle atmosphere.

The splitting of the wave flux indicates the pres-
ence of a partially reflecting obstacle to wave
propagation. Fig. 1b suggests that on day 15 this
obstacle is in the vicinity of 35°N and 25 km alti-
tude—almost precisely where maximum conver-
gence of wave flux was taking place on day 11 (Fig.
1a). Figs. 2a and 2b are cross sections of mean zonal
wind at two intermediate times (days 12 and 14).
We notice that a zero-wind line has moved through
the location in question, in consequence of the
northward encroachment of tropical easterlies (note
that the vertical scales are different in Figs. 1 and 2).

**Fig. 2.** Eulerian-mean zonal wind $\bar{u}(\phi, z, t)$ [m s$^{-1}$] for (a) day 12, (b) day 14, (c) day 20 and (d) day 22. The last two panels corre-
spond to the beginning of the second stage of the model warming, showing how $\bar{u}$ is changing just before the final, sudden reversal of
the polar westerlies. See paper I for $\bar{u}$ on days 0, 16, 18, 24 and 30. Contours are at intervals of 15 m s$^{-1}$. The velocity is held constant
at the artificial lower boundary, where its maximum value is 28.6 m s$^{-1}$ at 35°N, representing the tropospheric jet.
On day 15 the zero-wind line has reached the position indicated in Fig. 1b by the heavy curve.

The tendency for the wave flux to be deflected above and below the same location on which it had previously been converging strongly suggests that the ideas of nonlinear critical-layer theory might be relevant, since that theory predicts a tendency for zero-wind lines to turn into reflectors. We argue in Appendix B that nonlinear critical-layer reflection is indeed a qualitatively correct explanation of the splitting, despite the several violations of the premises of nonlinear critical-layer theory. The argument depends on the fact, which has not always been clearly recognized in the literature, that the simple wave, mean-flow coupling used in models like the present one which neglect higher zonal harmonics embodies essentially the same dynamical mechanism of nonlinear reflection as is found in "rational" models of nonlinear critical layers.

The most important departure from the premises of nonlinear critical-layer theory, for our purposes, is not the omission of higher zonal harmonics but rather the large-amplitude, transient state of the waves and mean flow. Critical-layer theory, both linear and nonlinear, assumes small wave amplitudes and long time scales. The effect of the large wave amplitude, and the substantial and rapid zonal-wind evolution in the present model, is to curtail evolution toward total reflection (for reasons again given in Appendix B). A partially reflecting state appears to be the most that is ever attained. The reflection is enough, however, to deflect more and more of the EP wave flux into high altitudes of the polar-cap region, and hence tend to focus it into a region of shrinking horizontal area and decreasing mean density. This focusing effect appears to play a significant role in bringing about the second and final stage of the model warming, as we shall see more clearly in the next section.

Figs. 1a–1c reveal another phenomenon which must be similarly significant for the model's behavior but whose causes are less evident. During the first stage of the simulation the overall magnitude of the wave flux increases to a greater extent than can be accounted for purely by focusing. For instance Figs. 1a–1c suggest, and a closer examination of the model output confirms, that there is a continual increase in $F_{\Phi}$ at the bottom boundary between days 11 and 24, even though the value of geopotential amplitude $|\Phi|$ imposed there is within 2.3% of its final steady value by day 11. Some of the increase in $F_{\Phi}$ seen in Figs. 1a–1c (and all of the increase at the lower boundary) must therefore be associated with an increase in the waves' westward phase tilt with height, by Eq. (2.11). Fig. 3a is a time-height section of $|\Phi|$ at 60°N. The lowest levels between 10 and 20 km show little sign of temporally increasing geopotential amplitude. Note, however, that there is a considerable increase higher up, at just the times and altitudes at which we would expect an increase due to the upward deflection and consequent focusing of the wave flux.

That the phase tilt does increase with time at the lowest levels is confirmed by Fig. 3b, which shows the longitude of one of the maxima in $\Phi$ at 60°N for days 11, 15, 22 and 24, as a function of $z$. The
corresponding buildup in $F_{\omega}$, beyond that which can be directly attributed to the focusing effect, could be due partly to changes in mean potential-vorticity gradients. Unlike $\widetilde{u}$, these are not constrained to be constant at the lower boundary. It could also be due to changes in wave structure of an intrinsically transient nature, directly induced by mean-flow deceleration. An obvious effect of the latter type occurs when the deceleration is rapid enough for the waves to be carried bodily with the mean flow, leading to the retrogression seen in the synoptic maps presented in I and familiar also from synoptic studies in the real atmosphere. Insofar as the retrogression rate increases with height, an increase in the westward phase tilt of the waves must be involved. This effect is almost certainly responsible, at least, for the suddenly increased phase tilt in the lower mesosphere between days 22 and 24 seen in Fig. 3b.

4. Zonal-mean deceleration described by the transformed Eulerian-mean equations

So far we have discussed mainly the early effects of mean-flow evolution on the pattern of planetary-wave propagation. In this and the following section we shall see how EP cross sections and related diagnostics provide a very direct view of the other half of the wave, mean-flow interaction problem, namely, how the waves change the mean flow— and in particular how they bring about the very large accelerations and temperature changes involved in the second and final stage of the warming, whose beginning is depicted in Figs. 2c and 2d.

The close relationship between F and mean-flow changes was first recognized by Charney and Drazin (1961), and is most clearly expressed by the transformed Eulerian-mean equations presented by Andrews and McIntyre (1976, 1978a). The quasi-geostrophic approximation to these equations, discussed by Edmon et al., can be written in the present coordinate system as

$$\partial \hat{u}/\partial t - f \hat{v}^* = \mathcal{F} + (\rho_0 a \cos \phi)^{-1} \nabla \cdot \mathbf{F}, \quad (4.1a)$$

$$f \hat{u}^* + \hat{u} \partial \theta_0/a = 0, \quad (4.1b)$$

$$(a \cos \phi)^{-1} (\hat{v}^* \cos \phi)_\phi + \rho_0^{-1} (\rho_0 \hat{w}^*)_\phi = 0, \quad (4.1c)$$

$$\partial \theta/\partial t + \hat{z} \hat{w}^* = \mathcal{H}, \quad (4.1d)$$

where

$$S = S(z) = g T_e^{-1} e^{-\kappa \pi t} = g \hat{T}/\partial T_e,$$

and where $\mathcal{F}$ and $\mathcal{H}$ are the Eulerian-mean friction and diabatic heating. The full ageostrophic forms are given in Appendix A. The residual meridional circulation ($\hat{v}^*, \hat{w}^*$) is defined by

$$\hat{v}^* = \hat{v} - \frac{1}{\rho_0} \frac{\partial}{\partial \hat{z}} \left( \frac{\rho_0}{\hat{z}} \frac{\hat{v}^*}{\hat{\theta}} \right), \quad (4.2a)$$

$$\hat{w}^* = \hat{w} + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\hat{v}^*}{\hat{\theta}} \right). \quad (4.2b)$$

Eqs. (4.1) comprise a complete set of equations for the mean state as described by the variables $\{\hat{u}, \hat{\theta}, \hat{v}^*, \hat{w}^*\}$. The property which makes these equations especially useful for understanding mean-flow evolution is that the term

$$D_F = (\rho_0 a \cos \phi)^{-1} \nabla \cdot \mathbf{F} \quad (4.3)$$

appearing on the right of Eq. (4.1a) is the only wave-induced forcing term in the equations. $D_F$ can be shown to equal the northward flux of quasi-geostrophic potential vorticity. Eq. (4.1a) shows that $D_F$ can be thought of as a zonally directed body force per unit mass acting on the mean state. The factor $(\rho_0 a \cos \phi)^{-1}$ arises because the true EP flux divergence $\nabla \cdot \mathbf{F}$ relates to angular momentum changes rather than directly to changes in $\hat{u}$.

Eqs. (4.1) have the same form as the standard Eulerian-mean equations, apart from the absence of a wave-induced eddy heat flux term on the right of (4.1d). Therefore we can immediately apply standard knowledge of how a zonally symmetric flow responds to a given force (Eliassen, 1951). It was shown in I and will be confirmed in the next section of this paper that the contribution induced by the diabatic heating $\mathcal{H}$ on the right-hand side of (4.1d) is of secondary importance only. Apart from that diabatic contribution, the residual circulation arises solely as the mechanism which prevents the force $D_F$ from putting the mean state out of thermal-wind balance. In the process it can only redistribute the effect of the force $D_F$.

An examination of the model output shows how this happens in our case. Fig. 4 illustrates the response of the mean zonal acceleration $\tilde{u}$ (heaviest curve) to the force $D_F$ at 40 km on day 15. Observe that $\tilde{u}$ tends to follow $D_F$, and also that the effect of $D_F$ is moderated by the positive values of $f \hat{v}^*$. Much the same pattern is found at other levels in the main, midlatitude region of negative torque $\nabla \cdot \mathbf{F}$ in Fig. 1b, from $-22$ km up to 42 km. Above and below that region, however, $f \hat{v}^*$ becomes negative. The net result is that the influence of the wave-induced force is extended vertically upward and downward, exactly as predicted by Eliassen’s theory.

The associated signature in the mass streamfunction is a two-cell structure with one cell extending below the region of maximum $|D_F|$ and the other above it. Fig. 5 shows the streamfunction for the residual circulation on day 15. The region within which $-D_F$ exceeds $10^{-5}$ m s$^{-2}$ is shaded. Contour values are chosen to show the two-cell structure despite the fact that it extends over many scale heights. The vertical scale of Fig. 5 differs from that of Fig. 1b. It is noteworthy that the lower cell extends well down into the virtual troposphere. Hori-
Horizontal redistribution of the effect of the force $D_F$ tends to be in a sense opposite to the vertical redistribution, again as one would expect from Eliassen's theory. Fig. 4 illustrates how the region where the sign of $\partial \tilde{u}/\partial t$ agrees with that of $D_F$ tends to be reduced in lateral extent by the effect of $f \tilde{v}^*$. To compare the description just given with the conventional Eulerian-mean description, note that the counterpart of $D_F$ in the conventional description is the eddy momentum flux convergence

$$D_M = -(a \cos^2 \varphi)^{-1} (\tilde{v}^* \tilde{u}^* \cos \varphi),$$

(cf. Eq. (A3a) of Appendix A. Fig. 4 illustrates the behavior of $D_M$ and the appropriate Coriolis term $f \tilde{v}$ [thinnest curves (the behavior is almost the same as in Holton, 1976, Fig. 6b)]. Both terms are considerably larger than $\tilde{u}$, tending to cancel each other, and they stand in no simple relationship to $\tilde{u}$. As is well known (Matsuno, 1971; Holton, op cit.) this more complicated situation is the result of wave-induced eddy heat fluxes forcing the mean state in addition to $D_M$ (mean state now being defined in terms of the conventional Eulerian-mean variables). The thin curves in Fig. 4 remind us that even the sign of $\tilde{u}$ can be difficult to guess from conventional diagnostics.

The simplification resulting from use of the transformed Eulerian-mean equations, namely, the existence of only one wave-induced forcing term in the

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**Fig. 5.** Mass streamfunction $\tilde{\chi}^*$ associated with the residual circulation on day 15. The warming can be regarded as an adiabatic response to the descending polar branch of this circulation. $\tilde{\chi}^*$ is defined such that $\delta \tilde{\chi}^*/\delta z = \rho_0 \tilde{\omega}^* \cos \varphi$ and $\delta \tilde{\chi}^*/\delta \varphi = -\rho_0 \tilde{\omega}^* \cos \varphi$. Contour values are to be multiplied by $\rho_0$, times $1 \text{ m}^2 \text{s}^{-1}$. Notice the non-uniform contour intervals. The shaded area represents values of $-D_F$ exceeding $10^{-4} \text{ m s}^{-2}$ (and reaching a maximum of $2.03 \times 10^{-4} \text{ m s}^{-2}$); see Eq. (4.3) for the definition of $D_F$. 

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mean equations, is extremely similar to that gained from thinking in terms of Lagrangian-mean dynamics (MN; Holton, 1980). In the latter case the wave flux involved is not the EP flux $F$, but rather the so-called radiation stress. Another conceptual gain from thinking in terms of $F$, or in terms of radiation stress, in place of the usual eddy heat and momentum fluxes, is that the artificial distinction between the effects of vertically and horizontally propagating planetary waves needs no longer be maintained. Whatever the net direction of wave propagation, as defined by $F$, the effect on the mean state is simply that a decelerating force is exerted on the mean zonal wind wherever $F$ converges. In particular, there is not much difference, from this viewpoint, between mean-flow deceleration by waves propagating purely vertically (as in the simplest model considered by Matsuno, 1971, Section 2, and by MN) and deceleration by waves propagating both vertically and poleward (as found by O’Neill and Taylor, 1979), apart from the stronger focusing into the polar cap in the latter case.

The transformed Eulerian-mean equations have the additional advantage of avoiding the severe
technical difficulties encountered by Lagrangian-mean descriptions for large-amplitude waves (Dunkerton, 1980; McIntyre, 1980a), some of which will be illustrated below.

5. Cross sections of $D_F$ and the irrelevance of critical-layer theory in the polar mesosphere

Fig. 6 presents cross sections of the effective zonal force per unit mass $D_F$ for days 16–26. Because of the factor $\rho_0(z)^{-1}$ in (4.3), these cross sections show events in the top half of the model which true EP cross sections like Fig. 1 cannot show without rescaling (Edmon et al., 1980, Section 7), owing to the small amounts of angular momentum involved. Compare for instance Fig. 6d with Fig. 1c, remembering the different altitude scales in the two figures. Fig. 7 shows the residual circulation for day 22, confirming that it has the same qualitatively simple form as before and extends over an even greater depth. Fig. 2d shows $\tilde{u}$ on day 22, which is just before the reversal of the polar westerlies.

The heavy curves in Fig. 6 are the zero-wind lines for each day. The development of the mid-stratospheric zero-wind line into a partial reflector of planetary waves, as discussed in Section 3 and Appendix B, takes place at successively higher altitudes during the first stage of the simulation (Figs. 6a–6c), albeit less and less strongly as the second stage is approached. Nevertheless, its influence clearly is important in bringing about the northward and upward movement of the principal maximum in $-D_F$, and hence a general trend toward larger magnitudes of $D_F$ because of the concentration of the EP wave flux into a smaller mass of the atmosphere as implied by the factor $(\rho_0 \cos \phi)^{-1}$ in (4.3). In addition, there is the increase in the EP wave flux from the bottom boundary, noted in Section 3. The whole process culminates in an explosive buildup in $D_F$ around day 24, almost like the crack of a whip (cf. Dunkerton, 1981).

The very great vertical spread of the contours of $D_F$ for day 22 seen in Fig. 6d is due to greatly increased wave transience. Conditions in the meso-

![Fig. 6. (Continued)](image)

![Fig. 7. As in Fig. 5 except for day 22; see Fig. 6d for the distribution of $D_F$ on day 22. Note the great vertical scale of the descending polar branch.)
spheric polar cap have reached an opposite extreme to those assumed by critical-layer theory; only thus can the mean flow change as rapidly, and in as deep a layer, as it does between days 22 and 24. Were wave transience less, mean rates of change slower, and the assumptions of critical-layer theory not quite so far from being satisfied, as was the case for the mid-stratospheric zero-wind line during the first stage, the model could be expected to develop non-linear reflection at the top of the polar-cap westerlies (again see the discussion in Appendix B), slowing down or preventing the onset of the warming. This is exactly what did happen in the β-plane model studied by Geisler (1974) when the wave input from below was sufficiently reduced. Fig. 6f shows that on day 26 the layer where $D_F$ is substantial has become much shallower, suggesting a return toward critical-layer conditions; and this is consistent with the slow descent of the zero-wind line from then on noted in I.

So far we have ignored the low-level, equatorward branch of the EP wave flux which was revealed by Figs. 1b and 1c, and the prominent negative feature in $\mathbf{V} \cdot \mathbf{F}$ which terminates it in the subtropics at 20 km altitude. This does not lead to much systematic change in $\tilde{u}$. The reasons are again the spherical geometry and the variation of mean density with height, this time working the opposite way round. In the low-altitude subtropics, the factor $(\rho_0 \cos \varphi)^{-1}$ in (4.3) means that $D_F$ and therefore the effect on $\tilde{u}$ is numerically quite small. The low-level feature in question is not, indeed, discernible in Fig. 6 without resorting to a much finer contour interval. This again illustrates just how important the $(\rho_0 \cos \varphi)^{-1}$ factor is—both in minimizing mean-flow changes at low altitudes and latitudes, and maximizing them when waves are focused into high altitudes and latitudes.

6. Lagrangian diagnostics, isentropic surfaces and cooling in the summer hemisphere

Up until now we have viewed the behavior of the model in terms of the braking effect of the waves on the Eulerian-mean zonal wind. The corresponding motion of isentropic surfaces relative to isobaric surfaces, implied by the thermal-wind relation, is well illustrated by the behavior of the 850 K isentropic surface. That surface is near the level of maximum warming at ~30 km. Its shape on day 26 is shown in Fig. 8 by means of contours of its pressure-altitude $z$. The projection of the surface onto a meridional cross section is shown for day 26 by the pairs of continuous curves appearing in Figs. 9a and 9b, which give the maximum and minimum values of $z$ on the surface at each latitude. The dashed curves in Fig. 9 give the shape of the same isentropic surface on day 0, when it is zonally symmetric.

The motion induced by the zonal wind reversal is
a bit like that of a see-saw in that descent of the isentropic surface near the pole is compensated by ascent in the tropics. This is another consequence of the model warming being sudden enough to be approximately an adiabatic process. The isentropic surface approximates a material surface dividing the atmosphere into upper and lower portions of constant mass. The mass of a vertical air column above such a surface is $g^{-1}$ times the pressure $p$ at the surface times the cross-sectional area $dA$ of the air column, and so the total mass above the surface is $g^{-1}$ times the global integral of $pdA$. Therefore,

$$ \iint pdA = \text{constant}, \quad (6.1) $$

where the integral is taken globally over the surface and $dA$ is the horizontal projection of a surface area element. The relation (6.1) reminds us in particular that any downward motion of the material surface relative to isobaric surfaces will be compensated by upward motion somewhere else.

This also tends to be true of the isentropic surface not only because the effects of diabatic heating are fairly small on the time scale considered, but also because the change in mass above the surface from diabatic cooling near the pole has opposite sign to that from diabatic heating in the tropics. Both statements can be verified from the discussion in I, or by noting the behavior of the two sets of air parcels shown in Fig. 9. On day 0 these sets are located at the black squares and distributed uniformly in the zonal direction; they are the same sets as those shown in Figs. 8, 9, 12 and 13 of I. The general tendency for both sets of parcels to remain near the isentropic surface is obvious. The greatest departures are at bottom left of Fig. 9b, where parcels which have descended deepest near the pole have been subject to the largest diabatic cooling rates, as discussed in I, and at top right of Fig. 9b where the highest parcels, at the tropical end of the "seesaw", have been systematically heated. A check on the mass lying above the model's 850 K isentropic surface shows that it has, in fact, changed only slightly between days 0 and 26. The mass change is equivalent to a globally averaged vertical displacement of the isentropic surface, relative to material surfaces, of only $-0.051$ km; this is much less than the amplitude of the "seesaw" motion evident in Fig. 9.

The property (6.1) can be expected to hold approximately for isentropic surfaces in real warmings. Irrespective of the dynamical details we can therefore expect to see a similar, seesaw like motion of isentropic surfaces relative to isobaric surfaces. That is why satellite radiometers commonly observe a drop in radiance over the whole of the tropics almost simultaneously with warmings in high latitudes (Fritz and Soules, 1970, 1972; Barnett, 1975; Quiroz, 1979a). The point, albeit elementary, is perhaps worth emphasizing in order to counter an impression sometimes met with that a more complicated description, in terms of Eulerian zonal means and eddy heat fluxes, needs to be invoked to explain such observations.

In the real stratosphere, the upward motion of isentropic surfaces in the tropics must overflow into the opposite hemisphere. Quiroz (1979a, p. 76 and Fig. 8) remarks on "the great spatial extent and apparent simultaneity" of the associated cooling, which is "coherently structured into the summer hemisphere as far poleward as 60° latitude". He suggests that this cannot be explained by any dynamical process. We digress briefly in order to show, on the contrary, how dynamical processes can easily explain what is observed.

We first ask how far the upward-moving end of the see-saw should extend into the summer hemisphere according to frictionless, adiabatic dynamics. The answer can be estimated in terms of the linear, low-frequency response of the hemisphere to zonally symmetric changes imposed at the equator. (We restrict attention to zonally symmetric changes on the assumption that planetary wave activity is relatively unimportant in the summer hemisphere.) The simplest self-consistent mathematical model neglects the summer-hemispheric zonal winds; the standard equations of tidal theory then apply (e.g., Holton, 1975). The response is computed by numerical solution of Laplace's tidal equation, the zonal wave number being set to zero and the frequency taken to be small in comparison with that of the semi-diurnal tide. Nonsingular behavior is imposed at the summer pole only. [It was verified, however, that the same computer program reproduced the global Hough functions tabulated in Longuet-Higgins (1968).]

Results for some of the graver equivalent depths are presented in Fig. 10. The curves represent the shapes of small vertical displacements of isentropic surfaces on the summer side of the equator, on an arbitrary amplitude scale, when the vertical structure of the change imposed at the equator has the modal structure corresponding to the equivalent depth shown. The actual vertical structure will involve a superposition of such modes, but it should be noted that the vertically sounding radiometers used for the observations see mainly the gravest modes, particularly the external (Lamb) and ducted internal mode shown, for instance, in Figs. 4a and 5a of Salby (1979). These have equivalent depths in the range 5–10 km. For such equivalent depths Fig. 10 shows that the zonally symmetric response penetrates all the way to the summer pole. The extent of the penetration, which on $\beta$-plane dynamics would be of the order of a Rossby radius of deformation, is greatly enhanced by the convergence of meridians on the sphere.
Fig. 10. Zonally-symmetric, adiabatic response of the opposite hemisphere to a small change imposed at the equator, according to the equations of tidal theory. Curves are labeled with values of the equivalent depth in kilometers. The amplitude scale is arbitrary. Each curve may be regarded either as the temperature change at a given pressure altitude, or as the vertical displacement of an isentropic surface relative to isobaric surfaces.

The speed of penetration that would occur if conditions at the equator were to change instantaneously would be of the order of the gravity-wave speed \((gh)^{1/2}\) for equivalent depth \(h\). For the gravest modes this is fast enough to explain the "near-simultaneity" noted by Quiroz and also by Fritz and Soules.

7. Generalized Lagrangian means, air-parcel dispersion and tracer transport

One way of defining "Lagrangian-mean motion" is as the motion of the apparent center of mass of a suitably chosen set of air parcels when projected onto the meridional plane. The black circles in Figs. 9a and 9b are the positions on day 26 of the computed centers of mass of the two sets of air parcels shown. The position of each circle relative to its corresponding square gives the net Lagrangian-mean vertical and meridional motion between days 0 and 26. Since the sets of air parcels were strung uniformly around their respective latitude circles on day 0, this Lagrangian mean is a case of the "generalized Lagrangian mean" (GLM) defined by Andrews and McIntyre (1978b), when the GLM theory is initialized relative to the undisturbed state on day 0.4

As was pointed out in I, the large spread in the positions of the individual parcels in Fig. 9b results mainly from two episodes of closed-streamline formation (the first of which is associated with the subtropical zero-wind line in the first stage of the warming, and the second of which occurs over the polar cap in the second stage and is the more realistic aspect of the model's behavior). To the extent that the first stage does not correspond to what happens in the real atmosphere (but see the discussion in Section 10), the latitudinal dispersion of parcels in middle and low latitudes may be overestimated by the model. Dispersion of air parcels is also apparent in Fig. 9a, for the set of air parcels initially at 60°N. Not all of this is irreversible, however. Fig. 9f of paper I shows that the set of parcels in question becomes confined between 37 and 54°N and between 30.2 and 31.6 km altitude by day 30. This contrasts with the behavior of the set shown in our Fig. 9b, for which most of the dispersion does seem to be irreversible (cf. I, Fig. 13f).

Despite the caveats just mentioned, these results do have a general bearing on past discussions of dispersion versus Lagrangian-mean motion as causes of tracer transport. They suggest that neither the Lagrangian-mean motion nor the dispersion about it is likely to be negligible. In the real stratosphere, parcel dispersion due to large-amplitude wave events probably does contribute significantly to wintertime tracer motions, even in a seasonal mean (A. J. Dyer, personal communication). Thus a correct viewpoint probably lies somewhere between the classical proposal by Brewer (1949) envisaging mean motion as the main cause of tracer motions in the stratosphere (see Dunkerton, 1978); and the complementary viewpoint emphasizing eddy-induced dispersion alone (e.g., Feely and Spar 1960). This point has also been made by Mahlman et al. (1980).

The north-south dispersion of air parcels is also an essential aspect of the dynamics, as is well known. As long as diabatic and frictional effects are only of secondary importance, as here, Ertel's potential vorticity behaves approximately like a passive tracer. Time-dependent parcel dispersion can therefore redistribute potential vorticity on isentropic surfaces and thereby change the Eulerian-mean state (Charney and Stern, 1962; Dickinson, 1969; see, also, Edmon et al. for a review of the ideas involved and their connection with our Figs. 1 and 6). It is therefore no surprise to find that the large changes in the mean state associated with a sudden warming should be accompanied by large air-parcel dispersion.

8. Lagrangian means and the divergence effect

A more systematic idea of Lagrangian-mean motions near the level of maximum warming at 30 km is given by Fig. 11, which again shows the day 0

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4 It is actually the Lagrangian mean of the vertical and meridional coordinates that is considered here, as in Kids (1977), rather than a true vector Lagrangian mean. The distinction is of some theoretical importance and has been discussed elsewhere (McIntyre, 1980a, Section 4).
and day 26 positions of the 850 K surface for comparison. The trajectories labeled A–H are center-of-mass motions up to day 26 for eight sets of air parcels initially at 30 km. The small crossbars mark the positions for days 10, 15, 20 and 25 on each trajectory. The labels A–H correspond to those used in Fig. 7 of I (which shows horizontal distributions of the same sets of parcels at particular times). The sets A–H are distributed uniformly around latitude circles at 5, 15, . . . , 75°N on day 0.

The trajectories show strong Lagrangian-mean descent in high latitudes and ascent in low latitudes. This feature is in qualitative accord with what we might have expected from MN’s model. It is also qualitatively like the residual circulations at 30 km implied by Figs. 5 and 7. Both the residual and Lagrangian-mean circulations are evidently quite different from the Eulerian mean meridional circulation for the same model simulation. For instance Fig. 3 of I shows strong Eulerian-mean ascent around 70°N and strong descent around 40°N on day 18. The only place where the Eulerian-mean circulation is anything like the residual and Lagrangian-mean circulations is the tropics, where wave activity is weak.

The detailed shapes of the mean trajectories in Fig. 11 show a complexity and irregularity not found in Lagrangian-mean motions calculated from simpler models like MN’s. Some of this is numerical, due inter alia to the finite number of air parcels whose positions are computed, but some of it is undoubtedly real, reflecting increasingly complicated air parcel distributions as the parcels disperse more and more widely (see Fig. 7 of I).

Another consequence of the dispersion is a strong “divergence effect” (McIntyre, 1973, 1980a; Andrews and McIntyre, 1978b; Uryu, 1979). This can arise for any kind of Lagrangian-mean velocity field—Stokes’ (1847) classical Lagrangian mean being no exception. Whenever wave amplitude is changing (and for this purpose wave amplitude is measured by air parcel displacement), Lagrangian-mean motions are not generally solenoidal and cannot be described by a streamfunction. In MN’s model the divergence effect is suppressed, except within the critical-layer singularity, because MN assumed steady, conservative waves outside the critical layer.

The essential phenomenon [familiar in connection with turbulent tracer transport (e.g., Rhines, 1973)] is that the center of mass of a dispersing set of air parcels will tend to move away from locations where eddy activity is effectively weak, such as rigid boundaries, toward locations where it is stronger. Now most of the divergence effect in our example is associated with north–south air-parcel displacements, as is also the case for the Lagrangian-mean motion calculated by Uryu (1979) for a growing Eady wave in a channel. Both examples involve quasi-geostrophic motions whose vertical parcel displacements are relatively insignificant for this purpose. In our hemispherical model the pole acts as one boundary and the equator as another; so while air parcels are dispersing horizontally the divergent part of the Lagrangian-mean motion will tend to be southward in polar latitudes and northward in tropical latitudes. This explains why Fig. 11 shows less northward Lagrangian-mean motion in high latitudes than we might expect at first sight from MN’s arguments, and more in lower latitudes. Note especially the long northward excursion for parcel set C, shown dotted. Parcel set E, in contrast, actually moves southward at first. In the second stage, when parcels nearer the pole are dispersing strongly, the northernmost set H makes a large southward excursion.

In MN’s analysis the northward flow and the divergence effect are both confined within a horizontal, linear critical layer. The divergence effect is then relatively negligible in the sense that although the velocities associated with it remain finite in the critical-layer limit, they occupy an infinitely thin layer and hence correspond to vanishing mass transport. By contrast, the systematic northward flow noted by MN carries a finite mass transport and involves infinite mean velocities in the critical-layer limit. In our simulation, both the northward flow and the divergent component occupy a deep layer (with the depth scales suggested by the regions of strongly negative $D_F$ in Fig. 6) and involve comparable velocities.

It is worth noting how the present results illustrate
the breakdown, at large wave amplitudes, of some of the definitions used in the GLM theory. Although there is no difficulty in defining the Lagrangian mean for a given set of air parcels, we must, in order to speak of a Lagrangian-mean velocity field, assign to each set of air parcels some reference position in height and latitude. The simplest choice is the mean (center-of-mass) position itself, and it is just this choice [albeit in the vector sense (see McIntyre, 1980a, Section 4)] which leads to the analytical power of the GLM theory which has proved so effective in other contexts (e.g., Leibovich, 1980). But for waves of large amplitude the divergence effect can become catastrophic in the sense that the center-of-mass positions of different sets of air parcels can run into each other (Dunkerton, 1980; McIntyre, 1980a). In such circumstances, reference positions are no longer uniquely assigned to sets of air parcels and the Jacobian determinant used in the GLM theory becomes singular. A coincidence of reference positions very nearly occurs for sets C and D on day 26 (note that D is displaced 3 km downward in Fig. 11 for clarity), and undoubtedly does occur for neighboring sets whose trajectories were not computed.

A related property of Lagrangian means is that the center-of-mass positions of all sets of parcels starting as zonal rings need not cover the entire latitude-height domain after finite-amplitude waves have developed; for instance, there can be a “gap” near the pole devoid of center-of-mass positions (McIntyre, 1980a). For reasons of symmetry and continuity this does not occur in the present case of a pure wave-2 warming but it can occur, for instance, when zonal wavenumber 1 is present.

9. Modified Lagrangian means: Results and discussion

One way of avoiding having to choose a particular initialization, or ensemble of air parcels, for defining a Lagrangian mean, is to use the “modified Lagrangian mean” discussed by Dunkerton (1980) and McIntyre (1980a,b). Unlike a true Lagrangian mean, the modified Lagrangian mean does not depend explicitly on the history of air-parcel motions, and can be computed directly from the instantaneous Eulerian fields produced by any model or even, in principle, from sufficiently accurate synoptic observations. It does, however, share some of the significant properties (and problems) of true Lagrangian means. It uses potential temperature, together with Ertel’s potential vorticity, as a way of following air-parcel motion approximately. In this section we use the model output to obtain a first idea of how modified Lagrangian means might behave in practice.

The definition of Ertel’s potential vorticity $P$ appropriate to our model equations is

$$P = \frac{1}{\rho_0} \times \left( f \frac{\partial \zeta}{\partial x} - \frac{(u \cos \varphi) \zeta}{a \cos \varphi} \right) \left( \frac{\partial u}{\partial z} - \frac{u_x \zeta}{a} \right), \quad (9.1)$$

The modified Lagrangian mean $\tilde{\psi}^M$ of any quantity $\psi$ may be defined as follows. Let $\theta$ be the value of the potential temperature for a given isentropic surface $S$. Denote by $\Delta S(P, \theta)$ that portion of the surface $S$ on which values of Ertel’s potential vorticity lie between $P$ and $P + \Delta P$. Fig. 12a illustrates how the 850 K isentropic surface is divided up into a number of such portions on day 10. On day 0 the $\Delta S$ are zonal rings. For small $\Delta P$ we define

$$\tilde{\psi}^M(P, \theta) = \int_{\Delta S} \left\{ \frac{\partial \theta}{\partial \rho} \right\}^{-1} \psi dA \int_{\Delta S} \left\{ \frac{\partial \theta}{\partial \rho} \right\}^{-1} dA \int_{\Delta S} \left\{ \frac{\partial \theta}{\partial z} \right\}^{-1} \rho_0(z) dA, \quad (9.2)$$

where $dA$ is the horizontal projection of an area element of $S$. This is a convenient form, for present purposes, of Eq. (25) of McIntyre (1980a), which states that $\tilde{\psi}^M$ is the average, weighted by mass, over a thin tube $C_M$ of air with potential vorticity lying between $P$ and $P + \Delta P$ and potential temperature between $\theta$ and $\theta + \Delta \theta$. It is the weighting by mass that gives rise to the factors $\left| \partial \theta / \partial \rho \right|^{-1}$.

This weighting implies that $\tilde{\psi}^M$ would equal the true generalized Lagrangian mean $\tilde{\psi}$ if $P$ and $\theta$ were exactly conserved at all times since initialization. Mass per unit length of the $C_M$ tube corresponds to number of parcels per unit length of the corresponding string of air parcels; wherever the parcels bunch together, the $C_M$ tube will be locally more massive. Eq. (9.2) is obtained from McIntyre’s Eq. (25) by letting $\Delta \theta$ and $\Delta P$ tend to zero in that order; the order does not matter since the cross-sectional shape of the tube $C_M$ is irrelevant provided the cross section is small. In the calculations, $\Delta P$ was taken as $\rho_s^{-1}$ times $6 \times 10^{-5}$ K m$^{-1}$ s$^{-1}$. Higher numerical resolution has no very obvious justification, if only because of the comparably coarse horizontal mesh size (5° latitude) used in the model. Our numerical method of computing $\tilde{\psi}^M$ is given in Appendix C.

Fig. 11 shows some trajectories of the points $\{\tilde{z}^M, \tilde{\zeta}^M\}$ for a set of $C_M$’s, more precisely $\Delta S$’s, on the 850 K isentropic surface. The significance of the solid and dashed portions of the curves will be indicated shortly. The labels “58–64”, etc., indicate the values of $P$ and $P + \Delta P$ used in (9.2), in units of $\rho_s^{-1}$ times $10^{-5}$ K m$^{-1}$ s$^{-1}$. As before, the small
crossbars mark days 10, 15, 20 and 25, except for the left-hand-most trajectory, which has no day-25 mark; it ceases to exist after day 23 because the corresponding $C_M$ tube disappears, a manifestation of non-conservation of $P$. Similarly, the trajectory labeled "52–58" ceases to exist after day 25. Fig. 13 gives more detailed information about $\tilde{\varphi}^M$ as a function of time, and Fig. 14 gives corresponding values of $\tilde{\vartheta}^M$ on selected days. Fig. 15 gives a more detailed view of $\tilde{\vartheta}^M$ as a function of time.

The general pattern of these results is broadly similar to that found earlier for the true generalized Lagrangian means. Because $P$ is not exactly conserved in the model, $\tilde{\vartheta}^M$ is not the same as the rate of change of $a\tilde{\varphi}^M$ (unlike $\tilde{\varphi}$ which does equal the rate of change of $a\varphi^M$). Nevertheless, Fig. 14 shows an overall tendency for southward mean motion (negative $\tilde{\vartheta}^M$) in high latitudes and northward motion in low latitudes, again symptomatic of the divergence effect.

Some of the differences between the true and modified Lagrangian means, as estimated here, and differences between $\tilde{\vartheta}^M$ and the rate of change of $a\tilde{\varphi}^M$, come from the fact that the model cannot
which initially zonal strings of air parcels are advected by the model motions. Many zonal wavenumbers are required to describe those shapes accurately, and so the same must be true of the corresponding $P$ distributions.

There is one aspect of the results not yet mentioned which would nevertheless be shared by accurate computations of modified Lagrangian means from higher resolution models. Fig. 12 illustrates how most of the $\Delta S$ cease to be simply connected as time goes on. Note, for example, how the shaded ribbon in Fig. 12 goes through three types of configuration. On about day 12 it first becomes multiply connected as in Fig. 12b. Around day 15 it breaks into three separate pieces, which subsequently become widely separated as illustrated in Fig. 12c. This again is a result of departures from strict conservation of $P$ and $\theta$, and such topological changes would undoubtedly occur even if dissipation were the only reason. The $10-16$ ribbon used in our calculations (initially just to the south of the shaded ribbon in Fig. 12, and overlapping it slightly) undergoes similar topological changes. The two smaller, poleward-moving parts of this ribbon have shrunk drastically in area by day 26; this is one reason why the $10-16$ trajectory in Fig. 11 turns southward while the C trajectory continues northward.

The times when the ribbons $\Delta S$ corresponding to the curves shown in Figs. 11, 13 and 15 first cease to be simply connected are indicated in those figures by the solid curves becoming dashed. The modified Lagrangian-mean latitude $\tilde{\varphi}^M$ can change rapidly during such events, because of the kinematics of

Fig. 13. Modified Lagrangian-mean latitudes $\tilde{\varphi}^M$ on the 850 K isentropic surface, as functions of time. The values of $\rho P$ and $\rho_a(P + \Delta P)$ used in Eq. (9.2) are indicated in units of $10^{-5}$ K m$^{-1}$ s$^{-1}$ below and above each curve, respectively; for example, the bottom curve is the average over values of $\rho P$ lying between $10 \times 10^{-5}$ K m$^{-1}$ s$^{-1}$ and $16 \times 10^{-5}$ K m$^{-1}$ s$^{-1}$.

Fig. 14. Modified Lagrangian-mean meridional velocities $\tilde{u}^M$ as functions of $\rho P$ on the 850 K isentropic surface, at various times (curves labeled in days). Note that the values differ from the rates of change of $a\tilde{\varphi}^M$. 
merging or splitting $\Delta S$'s. A good example is the sudden northward movement shown by the third and fourth curves from the top of Fig. 13 around day 20, contrary to the general trend associated with divergence of air-parcel ensembles away from the pole. These particular features are associated with fusion of $\Delta S$'s across the pole due to non-conservation, and not at all with material motions. Note that there is no hint of any corresponding reversal of the sign of $\hat{\omega}$ in the right half of Fig. 14.

In summary, our preliminary look at modified Lagrangian means, while useful as the only information to date giving an impression of their likely behavior and peculiarities for large-amplitude waves, has been distorted by the fact that $P$ is not handled conservatively by truncated spectral models like the present one. Despite this, the results do in some respects reproduce the qualitative behavior exhibited by true generalized Lagrangian means, for instance conforming roughly to what we expect to see from a consideration of the divergence effect and MN's arguments. Like true Lagrangian means the behavior of modified Lagrangian means is complicated when viewed in detail, the more so because of the splitting and merging of $C_M$ tubes which in the presence of dissipative processes is likely to occur at sufficiently large wave amplitudes, and in any case near critical lines. Such topological changes complicate the relationship between mean position and velocity, as is illustrated by close inspection of Figs. 13 and 14.

Modified Lagrangian means remain the only objective diagnostics, independent of choices of initialization or ensembles of air parcels, in terms of which the concept of Lagrangian mean may be expressed quantitatively for disturbances on a zonal flow. A more detailed and better assessment of their behavior must await the availability of fully nonlinear simulations with high numerical resolution in the zonal as well as in the latitudinal and vertical directions.

10. Concluding remarks, and another look at the observations

The model warming, particularly in its second stage (days 20–26), exhibits a number of fairly realistic features at which the new diagnostics have given us a closer look. These are the essentially adiabatic nature of the warming phenomenon; the suddenness of the mean zonal deceleration once the planetary waves are sufficiently focused into the high-altitude polar cap; the large depth scale of the mean deceleration at its peak, due to extreme wave transience and further extended downward and upward by the action of the residual circulation (the lower half of which produces the warming itself); the Eulerian-mean ascent and Lagrangian-mean descent in the polar cap; the see-saw like, zonally symmetric aspect of the displacements of material and isentropic surfaces (displacements which in the real atmosphere must extend far into the opposite hemisphere); and last but not least the fact that dispersion of air parcels, and a corresponding "divergence effect" in the Lagrangian-mean and modified Lagrangian-mean meridional motions, must accompany the drastic changes in the mean state brought about by the highly transient processes involved.

By contrast, the role of the mid-stratospheric zero-wind line in the first stage of the model warming (days 10–20) seems, at first sight, unlike anything that has been observed in real warmings. In particular, no evidence has been reported for a zero-wind line moving from the subtropics toward high latitudes, under the influence of wave 2 or otherwise. But focusing of the EP wave flux into the polar cap by some means or other is probably necessary in the real atmosphere in order to bring about the drastic changes observed in major warmings. Strong focusing, with the EP wave-flux arrows actually pointing in toward the pole, is definitely observed to take place in the real warmings studied by O'Neill and Taylor (1979) and Palmer (1981a, b). The mechanism or mechanisms by which the atmosphere achieves the focusing remain unclear, at least in the case of wave 2, and pose a central question for future research.

Several possibilities come to mind. One (following a suggestion by O'Neill and Taylor, 1979) is simply

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5 Note added in proof. Fig. 2a of Hirota and Sato (1969) perhaps could be interpreted as such evidence, however (A. J. Simmons, personal communication).
the poleward tilting of the EP flux arrows which is imposed from time to time by conditions in the troposphere. By Eqs. (2.2), an equatorward Eulerian-mean momentum flux and a poleward heat flux at the tropopause, will make the EP arrows tilt poleward there. One of us (CPFH) has tried a model run to test the idea. The result was negative. Despite a strong poleward tilt imposed at the lower boundary (other conditions being the same as in the present simulation), the initial tendency of the EP wave flux to turn equatorward and avoid the high-altitude polar cap seems very strong for wave 2 (cf. Matsuno, 1970), and the resulting simulation and its EP cross sections behaved almost exactly as before at 20 km and above. It remains possible, on the other hand, that O'Neill and Taylor's suggestion may be more relevant to other profiles of mean zonal wind or to the behavior of wave 1 and this, too, should be tested.

A second possibility is simply that there might exist mean profiles, not yet studied in any model simulation, which by themselves can produce significant focusing of wave 2 in the absence of zero-wind lines. At least three out of the five cases classified as typical "major midwinter warmings" by Labitzke (1977)—those of 1967–68, 1970–71 and 1972–73—strongly hint that the involvement of wave 2 in a major warming may in fact require preconditioning of the mean flow by the action of wave 1. The reason why the observations suggest wave-mean in preference to wave-wave interactions is that in the cases cited the geopotential amplitude of wave 1 (plotted against time together with wave 2 for 30 mb and 60°N in Labitzke's Fig. 1) is decreasing and sometimes well past its peak by the time the peak in wave 2 accompanying the warming occurs. A further and specially clear example of this is the February 1979 warming. The description in Quirroz (1979b) shows that it qualifies as a major warming in Labitzke's classification (op. cit., Section 4), in all respects except an unusually long delay, about four weeks, between the precursory wave-1 peak and the warming. It also is very clear, both from the description in Quirroz and from the EP diagnostics in Palmer (1981a), that when the warming did occur in late February it was caused by wave 2. Fig. 4 of Quirroz shows very low wave-2 amplitudes in the troposphere until well into February, suggesting that the warming occurred late simply because no wave-2 forcing was available earlier.

Therefore it does, in fact, seem reasonable to hypothesize that the essential role of the wave-1 peak, in the cases cited, was to change the mean flow in such a way as to focus any upward-propagating wave 2 into the polar cap. The hypothesis, in other words, is that the atmosphere underwent a "first stage" of mean-flow evolution by a mechanism different from the model's (and yet to be elucidated in detail) but which had similar consequences thereafter. It is certainly an observational fact that in the 1979 case the polar night jet was broad before the precursory wave-1 pulse occurred (Quirroz, 1979b, K. Labitzke, D. R. Pick, personal communication) and narrow afterward (ibid., Palmer, 1981a, Fig. 1a). The mean jet shown in Fig. 1a of Palmer (1981a) is in fact extremely similar, in latitudinal position and scale, to the jet in the polar cap shown in our Fig. 2c.

The main difference between Fig. 2c and Palmer's Fig. 1a is the fact that the latter shows no zero-wind lines. It would be of the greatest interest to test profiles like Palmer's in a model such as the present one. The first step could be to see if wave-2 forcing could produce a warming in one stage, resembling the event of late February 1979, before any zero-wind line developed equatorward of the jet. It is quite possible that at sufficiently large wave amplitude the difference between weak westertlies and zero westerlies would become inconsequential. The nonlinear reflection mechanism might then be able to come into operation and channel enough EP flux up the jet without any need for the mean wind to go through zero equatorward of the jet. It is also possible that there are jet profiles whose linear propagation properties would give sufficient channeling in any case. This could happen if mean northward gradients of quasi-geostrophic potential vorticity were sufficiently small on the south side of the jet—perhaps as a result of air-parcel dispersion due to the precursory wave-1 event. Closely related ideas have been independently proposed and discussed by Palmer (1981b).

It would be naive, of course, to think that wave-wave interactions can play no role at all, even if the model experiments just suggested support the hypothesis that mean-flow preconditioning is the crucial effect. Other experiments such as that of Hsu (1981) and Lordi et al. (1980) suggest that nonlinear interactions between waves 1 and 2 can be significant generally. It should be noted (A. J. Simmons, personal communication) that the strength of nonlinear interactions among different waves can, like the propagation and self-interaction of one wave, be sensitive to the profile of $\tilde{u}(x, z)$.

Yet another mechanism inviting consideration is linear interference (as opposed to nonlinear interaction) between stationary and traveling planetary waves of the same zonal wavenumber. This can produce transience and temporary focusing as noted by Palmer (1981a), which could conceivably force enough mean-flow change to initiate a positive-feedback process inducing further wave transience and perhaps prolonging the focusing. If traveling waves are significant, as indeed has been strongly suggested by a number of observational studies, models with more realistic lower boundary conditions like that of Schoeberl and Strobel (1980b) will be needed to study their effects.

EP cross sections, perhaps with the arrows re-
scaled as suggested in Section 7 of Edmon et al. (by a factor $e^{z/\bar{h}}$, say) may prove to be a helpful diagnostic tool for disentangling the various possibilities. We already know that the distribution and direction of the EP wave flux can vary in ways of which little hint is given by plots of geopotential amplitude alone. We know, moreover, that such variations are directly relevant to wave-induced changes in the mean state. A prime need, then, is to obtain a "comparative anatomy" of EP cross sections, isolating the contributions to $F$ and $D_F$ from different zonal harmonics and noting the circumstances under which the EP wave flux is focused into the polar cap. This, of course, should be done for hypothesis-testing model experiments as well as for observed warmings.

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APPENDIX A

Ageostrophic Forms of the Equations in Log-Pressure Coordinates

The ageostrophic forms of the equations and formulas used in this paper are collected here for reference. They are equivalent to the results presented by Andrews and McIntyre (1978a), specialized to perfect-gas thermodynamics and with $p_s e^{-z/\bar{h}}$ substituted for $p$. They depend neither on the Boussinesq approximation as in Andrews and McIntyre (1976), nor on Holton's (1975) thermodynamic approximations as in the work of Boyd (1976), who presented related results for the case of exponentially growing or decaying waves.

The divergence of $F$ is given by (2.3) with

$$F_{(\psi)} = \rho_0 a \cos \varphi \frac{v' \theta'}{\theta_z} - \varphi u'$$

(A1a)

$$F_{(\omega)} = \rho_0 a \cos \varphi$$

$$\times \left[ \left( f - \frac{\bar{u} \cos \varphi}{a \cos \varphi} \right) \frac{v' \theta'}{\theta_z} - \varphi w'u' \right]. \quad (A1b)$$

The residual meridional circulation is still defined by (4.2), and the transformed Eulerian-mean equations are:

$$\frac{\partial \bar{u}}{\partial t} + \left( \bar{u} \cos \varphi \right) \frac{\partial}{\partial \varphi} - f \varphi + \bar{u} \bar{w}$$

$$= 0$$

$$+ \frac{1}{\rho_0 a \cos \varphi} \nabla \cdot \mathbf{F}, \quad (A2a)$$

$$f + 2a^{-1} \bar{u} \tan \varphi \bar{u} + S \bar{\beta}_o \bar{a}^{-1}$$

$$= \frac{\partial^2 \bar{v}^*}{\partial \varphi \partial t} + \frac{\partial \bar{v}^*}{\partial \varphi}$$

$$\left( a \cos \varphi \right)^{-1} \left( \bar{v}^* \cos \varphi \right) + \rho_0 a \bar{w}^* = 0, \quad (A2c)$$

$$\frac{\partial \bar{v}^*}{\partial \varphi} + \frac{a}{\rho_0 a \cos \varphi} \left( \bar{u} \bar{w} \bar{v}^* \right)$$

$$= \varphi - \rho_0 a \cos \varphi \frac{\partial}{\partial \varphi} \left( \frac{\bar{v}^*}{a \theta_z} \bar{v}^* \theta' + \bar{w}^* \theta' \right). \quad (A2d)$$

Here $\left[ \right]$ in (A2b) denotes the expression within square brackets on the right of (3.7b) of Andrews and McIntyre (1978a), plus a term $-a^{-1} \bar{u}^2 \tan \varphi$ which they inadvertently omitted, as well as terms involving $\bar{v}^2$ and $\bar{v} \bar{w}$ which they denoted collectively by $O(a^4)$. The right-hand side of (A2b) is usually very small in meteorological applications, and cannot in any case do more than to shift the thermal wind balance of the mean state. The term within the square brackets of (A2d) is proportional to the component of eddy heat flux parallel to the mean potential temperature gradient, and is zero for steady, conservative, linear waves (Andrews and McIntyre 1976, 1978a; Plumb, 1979).

It is straightforward to verify that substitution of (4.2) into the standard Eulerian-mean primitive equations does, in fact, give Eqs. (A2); the Eulerian-mean momentum and potential temperature equations in the present log-pressure coordinates are.

$$\frac{\partial \bar{u}}{\partial t} + \left( \bar{u} \cos \varphi \right) \frac{\partial}{\partial \varphi} - f \bar{v} + \bar{u} \bar{w}$$

$$= \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left( \cos^2 \varphi \bar{v}^* u' \right)$$

$$- \rho_0 a \cos \varphi \frac{\partial}{\partial \varphi} \left( \rho_0 \bar{w} \bar{u}^* \right). \quad (A3a)$$
\[ \frac{\partial \tilde{\theta}}{\partial t} + \frac{1}{a} \frac{\partial \tilde{\theta} \tilde{v}}{\partial z} + \tilde{\theta} \tilde{\omega} = \tilde{\omega} - \frac{1}{a} \frac{\partial}{\partial \varphi} \left( \cos \varphi \varphi \tilde{\omega} \right) \]

\[ - \psi^{-1} \frac{\partial}{\partial \varphi} \left( \rho_0 \psi \tilde{\omega} \right) . \] (A3d)

It is also straightforward to verify by standard order-of-magnitude arguments that the bracketed term on the right of (A2d) is negligible if the eddies satisfy quasi-geostrophic scaling. Likewise negligible are the terms in \( \tilde{\omega} (\tilde{v} \cos \varphi) \), \( \tilde{w} \tilde{\omega} \), and all terms involving \( \tilde{u} \) or \( \tilde{w} \) in (A1). These are the terms neglected to obtain the equations used in Sections 2 and 4.

**APPENDIX B**

**Nonlinear Reflection from Critical Layers**

At first sight, the analytical theory of nonlinear critical-layer behavior seems unlikely to be applicable to waves of such large amplitude as here. The theory depends explicitly on a formal assumption of small wave amplitude, and hence small zonal wind changes \( \Delta \tilde{u} \) taking place over time scales much longer than typical advection time scales for a stationary planetary wave. On the other hand, the results of a numerical experiment reported by Bélard (1978) suggests that the qualitative features which concern us may carry over to wave amplitudes larger than might be expected.

Bélard’s results are consistent with a key insight into nonlinear critical-layer behavior suggested by the analytical theory, most clearly by its recent development due to Stewartson (1978) and Warn and Warn (1978). The analytical theory states that the development of wave reflection from a nonlinear critical layer of the Rossby-wave type can be regarded as a direct consequence of vorticity advection. The flow within the closed-streamline regions of the critical layer leads to a coiling-up of the absolute vorticity distribution. For baroclinic Rossby waves we may expect that the same insight will hold good provided we think in terms of the distribution of quasi-geostrophic potential vorticity, or of Ertel’s potential vorticity on isentropic surfaces (cf. Charney and Stern, 1962). The connection between wave reflection and the vorticity distribution is discussed in simple terms by McIntyre.\(^6\)

The Z shaped features distinguished by shading in Fig. 12a above are found, on examination of the model output, to be roughly centered on the two regions of closed streamlines around 30°N (see also Figs. 5a and 12a of I, and recall that streamfunction \( = \tilde{f}^{-1} \Phi \)). The Z shapes result from the twisting round of the potential-vorticity distribution by the closed-streamline flow. They strongly resemble vorticity distributions found in the analytical theory when nonlinear reflection is just beginning. A good example is that shown in Fig. 2c of Warn and Warn (1978).

However, Figs. 12a and 12b show that the resemblance diminishes before the shaded “ribbon” can coil up sufficiently to look like a vorticity distribution giving perfect reflection, such as that shown in Warn and Warn’s Fig. 2d. In our case the mean-flow evolution is much stronger than in theirs, resulting in rapid northward migration of the centers of the closed-streamline regions, as discussed in paper I. This keeps the gross features of the potential-vorticity pattern somewhere between that of Fig. 12a and the perfectly reflecting, symmetrically coiled pattern in Warn and Warn’s Fig. 2d. Thus a qualitative consideration of the nonlinear reflection mechanism leads to the same conclusion as the direct evidence of Fig. 1b. Both strongly suggest that the zero-wind line in the middle stratosphere has become a partial reflector by day 15.

This conclusion seems broadly consistent with the fact that there is a noticeable slowing down, between days 14 and 18, of the rate at which the easterlies encroach northwards. It should be noted, of course, that even if the zero-wind line were to become a perfect reflector at some level the mean flow could still change there because of the action of a “residual circulation” induced higher up, as discussed in Section 4.

There is one point of some theoretical interest which our discussion has glossed over so far and which has not always been clearly appreciated in the literature on nonlinear critical layers. The present numerical model does not include the higher harmonics of zonal wavenumber 2, and so cannot actually describe all the details of potential vorticity distributions like those involved in nonlinear critical layers. It thus resembles the quasi-linear critical layer model of Geisler and Dickinson (1974) more than the “rational” critical layer models cited above. As discussed in I and in Section 9 of the present paper, this is one reason why the contours in our Fig. 12 do not resemble more closely the shapes of material lines displayed in Figs. 12 of paper I.

It can nevertheless be shown that the onset of nonlinear reflection predicted by the “rational” critical layer theory of Stewartson and Warn and Warn is surprisingly well imitated by models neglecting higher harmonics, even as far as the overreflecting stage which follows the attainment of per-

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fect reflection when wave amplitude is small (Geisler and Dickinson, 1974, Fig. 6; Warn and Warn, 1978, Fig. 3a). Fig. 2 of McIntyre gives a quantitative comparison for Warn and Warn’s case. The reason for the good performance by quasi-linear models in this respect is that they can imitate to a sufficient degree the most important dynamical effect of the coiling-up of vorticity, namely, the reversal of the northward absolute vorticity gradient within the closed streamline regions. This reverses the sign of vorticity advection by the meridional velocity \( v' \). Quasi-linear models like Geisler and Dickinson’s bring this about simply by reversing the zonal-mean absolute vorticity gradient \( \beta - \bar{u}_{yy} \), where \( \beta \) is the planetary vorticity gradient. In a thin critical layer \( \beta - \bar{u}_{yy} \) can reverse even though the change in \( \bar{u} \) itself remains small, as Geisler and Dickinson noted. Both \( \bar{u} \) and \( \beta - \bar{u}_{yy} \), in fact, behave in very much the same way in the “rational” as in the quasi-linear models, up to the first onset of perfect reflection.

The discussion would be incomplete without mentioning the complications in the real atmosphere which might be introduced by instabilities of the nonlinear critical layer (McIntyre). These instabilities are suppressed both by truncated spectral models of the present type, and by “rational” models of nonlinear critical layers. It appears from calculations by Killworth and McIntyre to be reported elsewhere that the instabilities would tend to weaken nonlinear reflection, especially at small wave amplitudes and low latitudes.

**APPENDIX C**

**Method of Computing Modified Lagrangian Means**

The modified Lagrangian means presented in this paper are computed using Eq. (9.2). It is clear from this equation that, in order to compute \( \psi^m \), one needs to know the values of \( P, \psi, \partial \psi / \partial z \) and \( \rho_0 \) (which is equal to the constant \( \rho_0 \) times \( e^{-z/H} \)) on a given isentropic (e.g., \( \theta = 850 \, \text{K} \)) surface.

Since \( \theta \) is equal to \( T \exp(\kappa z/H) \) and increases rapidly with \( z \) in the stratosphere-mesosphere region, it is reasonable to assume that, between two grid points in the vertical direction, \( \ln \theta \) is a linear function of \( z \). We note that this assumption would be perfect if the atmosphere were isothermal. On the basis of this assumption, we obtain the values of \( \theta \) and \( \partial \theta / \partial z \) on a \( 10^\circ \) longitude \( \times 5^\circ \) latitude grid on the isentropic surface as follows. For any given \( \lambda \) (longitude), \( \varphi \) (latitude) and \( \theta \), we first locate the index \( k \) such that \( \theta_k \leq \theta < \theta_{k+1} \), where \( \theta_k \) denotes the value of \( \theta \) at the grid point \( (\lambda, \varphi, \theta_k) \). We then have

\[
z(\lambda, \varphi, \theta) = z_k + \frac{\ln(\theta) - \ln(\theta_k)}{\ln(\theta_{k+1}) - \ln(\theta_k)} \Delta z,
\]

Here \( \Delta z \) denotes the vertical grid length. The values of \( u, \partial u / \partial \varphi \) (at constant \( z \)) and \( \partial u / \partial z \) on the isentropic surface are then computed from

\[
u(\lambda, \varphi, \varphi) = u_k + \frac{u_{k+1} - u_k}{\Delta z} (z - z_k),
\]

\[
u(\lambda, \varphi + \Delta \varphi, \varphi) = u(\lambda, \varphi + \Delta \varphi, z) - u(\lambda, \varphi, \varphi) = \frac{u_{k+1} - u_k}{2 \Delta \varphi},
\]

In the above equations \( z \) denotes \( z(\lambda, \varphi, \theta) \). The values of \( u(\lambda, \varphi, \pm \Delta \varphi, z) \) in (C4) are computed by linear interpolation from \( u(\lambda, \varphi \pm \Delta \varphi, z_k) \) and \( u(\lambda, \varphi, \pm \Delta \varphi, z_{k+1}) \). Eq. (C5) is obtained by assuming that \( u \) is a linear function of \( z \) for \( z_k \leq z < z_{k+1} \).

Eqs. similar to (C3)–(C5) are used to compute \( v, \partial u / \partial \lambda, \partial u / \partial \varphi, \partial v / \partial \lambda \) and \( \partial v / \partial \varphi \) as functions of \( \lambda, \varphi \) and \( \theta \). We then compute \( P(\lambda, \varphi, \theta) \) by Eq. (9.1).

After obtaining the values of \( z \) (which gives \( \rho_0 \), \( \partial \theta / \partial z \), \( P \) and \( \psi \) which denotes one of the following parameters: \( \varphi, z, \Phi, u, v \) and \( w \)) on the \( 10^\circ \times 5^\circ \) grid on the \( \theta \) surface, we use linear interpolation to obtain their values on a \( 1^\circ \) longitude \( \times 5^\circ \) latitude grid. Now \( \Delta S \) in Eq. (9.2) denotes the portion of the isentropic surface on which values of \( P \) lie between \( P_1 \) and \( P_1 + \Delta P \), where \( P_1 \) and \( \Delta P \) are numbers previously chosen. Therefore, we can obtain a good approximation to \( \int_{\Delta S} (\partial \psi / \partial z)^{-1} \rho \psi dA \) by summing up the values of the integrand multiplied by \( \cos \varphi \) at those grid points, on the finer grid, where \( P_1 \leq P < P_1 + \Delta P \). Similarly, we can obtain \( \int_{\Delta S} (\partial \psi / \partial z)^{-1} \rho dA \). Substituting these results into Eq. (9.2) gives the desired result \( \psi^m \).

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