Eigenfrequencies and Horizontal Structure of Divergent Barotropic Instability Originating in Tropical Latitudes

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ABSTRACT

Instabilities arising on a latitudinally sheared mean flow fall into one of at least two classes: inertial instabilities associated with a reversed potential vorticity and barotropic instabilities associated with a reversed meridional gradient of potential vorticity. Both types of instability are described by the generalized Laplace tidal equation, a horizontal structure equation that explicitly includes the effect of horizontal divergence on the disturbances. The effect of horizontal divergence on barotropic instability has not been extensively studied. A systematic investigation of the eigenfunctions of the generalized Laplace tidal equation for monotonic mean zonal wind profiles having a single, narrow region of reversed vorticity gradient in tropical latitudes reveals that, in the limit of low planetary zonal wavenumber, the modes of barotropic instability bifurcate into weakly divergent modes of hemispheric scale, and strongly divergent, "internal" modes trapped about the source region, i.e., equatorially trapped. Disturbances in the second category penetrate into the deep tropics—the side of the critical latitude with positive intrinsic frequency—as a Kelvin wave type of behavior not previously seen in this context.

These results suggest, first, that hemispheric barotropic instability need not be purely nondivergent. In fact, the growth of weakly divergent modes is preferred. Their equivalent depth is similar to that of free neutral modes of the homogeneous vertical structure equation. Second, the existence of equatorially trapped divergent barotropic instability may be of interest in the tropical troposphere and mesosphere. The equatorial amplitude of these disturbances can be significant, and their frequency, which is generally less than that of a dry Kelvin wave, is determined by a critical latitude in the region of reversed vorticity gradient.

1. Introduction

The horizontal structure of linear oscillations in a uniformly rotating, stratified atmosphere is governed by the Laplace tidal equation

\[ \mathcal{L} \{ \phi \} = \epsilon \phi \]

(1.1)

where \( \phi \) is geopotential and \( \epsilon \) is the eigenvalue or separation constant, inversely proportional to equivalent depth (Flattery 1967; Longuet-Higgins 1968). If the mean zonal wind varies in latitude but not in height, or is slowly varying in height (Boyd 1978), a generalized version of (1.1) may be derived (see Appendix). Latitudinal shear alters the stable modes of (1.1), but more importantly, this shear can also give rise to unstable modes if certain criteria are met. The known instabilities fall into one of two classes: (i) inertial instabilities associated with a reversed potential vorticity, modes which may be zonally symmetric (Dunkerton 1981; Stevens 1983) or nonsymmetric (Boyd and Christidis 1982; Dunkerton 1983a; Stevens and Ciesielski 1986); (ii) barotropic instabilities associated with a reversed meridional gradient of potential vorticity (Dickinson and Clare 1973; Lindzen and Tung 1978; Pfister 1979; Hartmann 1983; Dunkerton 1987). Barotropic instabilities are often assumed to be horizontally nondivergent (\( \epsilon = 0 \)) but, as we shall see, this is an unnecessary and even undesirable restriction to the general equation (1.1).

An assessment of the relative importance of inertial and barotropic instability in the atmosphere is guided by the respective instability criteria which, on the one hand, make inertial instability difficult to achieve; a reversed potential vorticity requires a departure from conservative motion (Hoskins 1974) or cross-equatorial transport of angular momentum (Dunkerton 1981). On the other hand, a barotropically unstable flow can be set up by a simple conservative redistribution of the potential vorticity field (e.g., Killworth and McIntyre 1985; Haynes 1985). A second consideration is that inertial instability, as a "parcel" instability, is largely confined to the region of anomalous potential vorticity, and is characterized by small vertical wavelength, making it susceptible to scale-dependent damping. In contrast, barotropic instability involves Rossby waves (e.g., Lindzen and Tung 1978) which, if allowed to propagate, can occupy a significant fraction of the sphere (Dunkerton 1987). The source region need not be extensive in size; frequency selection is determined by a critical latitude within the region of

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reversed potential vorticity gradient (Dickinson and 
Clare 1973; Lindzen and Tung 1978).

The existence of a class of horizontally divergent
instabilities ($\epsilon > 0$) in a barotropically unstable mean
flow has not been widely recognized (Hartmann 1983;
Stevens and Ciesielski 1986); in the separable problem
these instabilities are baroclinic, that is, possess oscil-
Iatory vertical structure, but they arise from the lati-
tudinal shear of the mean zonal wind. Their horizontal
structure is in some respects similar to that of nondi-
vergent barotropic instability (critical latitude, juxta-
position of propagating and evanescent regions, etc.).
To avoid confusion with baroclinic instabilities arising
from the vertical shear or lower boundary condition,
the abbreviated term "divergent barotropic instability"
will be applied to this class of solutions (1.1). 1

The purpose of this paper is to investigate unstable
solutions of (1.1) for nonzero but real $\epsilon$, representing
horizontally divergent modes of instability on the
sphere. With a little care, it is straightforward to con-
struct barotropically unstable mean flow profiles near
the equator which are inertially stable—the opposite
situation from that considered by Boyd and Christidis
(1982) and Dunkerton (1983a). 2 In this way, a clear
separation can be maintained between inertial and
barotropic instability.

A practical motive for this study is to further explore
the role of barotropic instability in the tropical trop-
osphere and mesosphere. In both regions, the neces-
sary condition for instability is satisfied in the equatorial
flank of the winter westerly jet. Although zonal vari-
ations of the basic state may help determine the lon-
gitude of instability (Simmons et al. 1983), it will be
pedagogically useful to first examine the instability of
zonally symmetric basic states. By restricting attention
to a few monotonic profiles of mean zonal wind with
a single, narrow region of reversed vorticity gradient,
it will be possible to confirm the mechanisms respon-
sible for frequency selection and horizontal structure,
and to note the effects of horizontal divergence. It will
be shown that for shears representative of the tropical
troposphere, unstable modes exist which, in the limit
of small zonal wavenumber, bifurcate into weakly di-

1 This hybrid term is a semantical compromise; it should be kept
in mind that "divergent" pertains to the horizontal velocity field of
the disturbance, and "barotropic" refers to the mean zonal wind as
the source of the instability. No limitation is placed on the magnitude
of horizontal divergence, nor is any balance assumption made. In
this respect divergent barotropic instability differs from the weakly
divergent quasi-geostrophic instability discussed extensively in the
literature of the 1960s (e.g., Pedlosky 1964, and references therein).
In comparison, little theoretical work has been done on the general
case. Ripa (1983), for example, derived a generalized "Fjortoft the-
orem" for the shallow water system, a modified form of which applies
directly to the equation set in the Appendix.

2 Stevens and Ciesielski (1986) examined sech^2 jet profiles, which
were inertially and barotropically unstable.

2. Solution methods

The form of Eq. (1.1) is derived in the Appendix
[see (A25) and (A26)]. The geopotential equation
contains "apparent" singularities at the inertial lati-
itudes ($\Delta = 0$) and a first-order singularity at the critical
latitude ($\phi = 0$). Obviously, if the frequency lies off
the real axis, there are no singularities of either type,
and it is straightforward to solve (1.1) using conven-
tional methods.

a. Iteration over $\omega$, $\epsilon$

A shooting method is perhaps the most efficient way
to solve the second-order boundary value problem:

$$A_2 \phi_{\mu\mu} + A_1 \phi_{\mu} + A_0 \phi = 0 \quad (2.1a)$$

$$\phi(\pm 1) = 0 \quad (2.1b)$$

by an accurate guess for the eigenfrequency (or $\epsilon$) is
available. Even without a good first guess, it is routine
to search a large number of combinations ($\omega, \epsilon$) until
an eigenmode is located. Then, the trajectory of this
mode can be followed through the parameter space.

Equation (2.1a) [derived from (A26)] was written in
finite difference form and solved with the tridiagonal
algorithm, working from both poles into equatorial
latitudes and applying matching conditions there. 200–
2000 grid points were used, equally spaced in $\mu$
$= \sin(\text{lat})$ ($\Delta \mu = 0.001$–0.01). All variables were as-
signed as double precision complex, to ensure accuracy
in the calculation of continued fractions.

b. Time-dependent solution (most unstable mode)

To help validate the solutions obtained with iteration,
a time-dependent model was derived from Eqs. (A10)–(A12), making the equatorial beta-plane ap-
proximation. Implicit time-stepping was used, with no
a priori assumptions about time-dependence:

$$-\dot{\omega}u + (\tilde{u}_y - \beta \tilde{y})v + k\phi = -\omega u_n \quad (2.2a)$$

$$+\dot{\omega}v + \beta v + \phi_y = +\omega v_n \quad (2.2b)$$

$$-\epsilon \omega \phi + ku + v_y = -\epsilon \omega \phi_n. \quad (2.2c)$$
The symbol \( \omega \) has a different meaning here than elsewhere in this paper, viz.,
\[
\omega = \frac{2i}{\Delta t}
\]  \hspace{1cm} (2.3)
and the subscript \( n \) indicates the value at the previous time step. The next value is
\[
\phi_{n+1} = 2\phi_n - \phi_n
\]  \hspace{1cm} (2.4)
and so on. The implicit method requires no diffusion for numerical stability, but is able to isolate only the most unstable mode (if one exists). The equatorial beta-plane is valid only for disturbances trapped in the vicinity of the equator. Equations (2.2a–c) were combined to yield a geopotential equation, which was solved in a channel extending from \( 40^\circ S \) to \( 60^\circ N \), with 150 grid points equally spaced in dimensional latitude \( y = a\theta \).

**c. Comments**

These two methods should not yield identical results because of the equatorial beta-plane approximation used in the second method; nevertheless, it will be shown that suitably trapped solutions exist for large \( \epsilon \), and these modes yield good agreement between the two methods.

In what follows, we restrict attention to real \( \epsilon \) and complex \( \omega \) (\( \text{Im} \omega > 0 \)). Boyd (1981, 1982) discusses the general case (complex \( \epsilon \)) for near-neutral modes. Derivation of such modes not only adds a degree of freedom to the problem (expanding the parameter domain to be searched) but seems to demand accuracy near the critical latitude singularity. This difficulty can be avoided by restricting attention to unstable modes for which the eigenfrequency lies off the real axis by a finite and resolvable amount.\(^3\)

**3. Results**

**a. Mean flow profiles**

The climatological flows of the winter troposphere and mesosphere have two features in common; both jet streams are centered between the subtropics and midlatitudes, and both satisfy the necessary condition for barotropic instability at low latitudes. (Zonal variations of the jet imply that certain regions are more unstable than others: Simmons et al. 1983.) The mesospheric jet may also be unstable on its poleward flank (Pfister 1979; Hartmann 1983). There are, of course, several other differences between the two regions. The subtropical mesospheric jet is much stronger than the tropospheric jet stream (Dunkerton and Delisi 1985).

Cross-equatorial shear exists in the mesosphere at the solstices, which may be inertially unstable (Dunkerton 1981, 1983a; Hitchman et al. 1987). In the tropical troposphere the cross-equatorial shear is much smaller; the 200 mb flow is roughly symmetric about the equator. However, the latitudinal shear separating the tropical flow from the jet stream is considerably stronger in the winter hemisphere (Newell et al. 1974; Arkin and Webster 1985; Rosenlof et al. 1986; Liebmann 1987).

For instability calculations it is desirable to choose a mean flow that is realistic in the region of interest, but simple enough to study theoretically without undue complications from other effects. Although the real atmosphere may exhibit different kinds of instability at other latitudes, an attempt to capture all of these effects at once would not only introduce other unstable modes into the problem, but would also make the interpretation of the modes we want to study more difficult. Therefore, we will limit our discussion to monotonic profiles of the form
\[
\bar{\sigma} = \frac{U}{2\Omega a} \left[ 1 + \tanh \frac{\theta - \theta_0}{\theta_1} \right] / 2.
\]  \hspace{1cm} (3.1)

In this section, the constants \( U, \theta_0, \) and \( \theta_1 \) will be chosen in such a way that the flow is inertially stable everywhere, and barotropically unstable within a narrow region in the winter hemisphere, just north of the equator. Two combinations will be examined:

\[
\begin{align*}
U &= 50 \text{ m s}^{-1} \\
\theta_0 &= 20^\circ \\
\theta_1 &= 5^\circ \\
U &= 40 \text{ m s}^{-1} \\
\theta_0 &= 13^\circ \\
\theta_1 &= 6^\circ \\
\end{align*}
\]

Case I

Case II.

These profiles are shown in Fig. 1, along with the mean vorticity and vorticity gradient. Note that both profiles are inertially stable (vorticity is the same sign as \( f \)). The reversal of vorticity gradient is stronger in Case I than in Case II, but the region of reversed gradient is closer to the equator in Case II. These profiles are fairly representative of the 200 mb tropospheric flow in the Pacific region (as far as the location of reversed gradient is concerned) according to the observational studies quoted above (cf. Simmons 1982). The hyperbolic tangent profile, by definition, is not necessarily stable according to Ripa's (1983) theorem.

**b. Eigenfrequencies for Case I**

Figure 2 shows the nondimensional eigenfrequency as a function of \( \epsilon^* \) for Case I, covering the geophysically...
Fig. 1. Mean flow profiles for Case I (solid) and Case II (dashed): (a) mean zonal wind divided by \( \cos \theta \); (b) nondimensional vorticity; (c) nondimensional vorticity gradient.

Fig. 2. Eigenfrequencies of divergent barotropic instability for Case I: (a) Re\( \sigma \); (b), (c) Im\( \sigma \). Curves are labeled with planetary zonal wavenumber \( s \).
relevant range of eigenvalues from $\epsilon^* = 1$ to $\epsilon^* = 0.001$. The equivalent depth $h$ is defined as

$$\epsilon^* = \frac{(2\Omega a)^2}{gh}.$$  \hspace{1cm} (3.2)

Free modes of the homogeneous vertical structure equation have

$$h = \frac{7}{5} H$$ \hspace{1cm} (3.3)

(Andrews et al. 1987, p. 172). For this choice of $h$, $\epsilon^* \sim 10$. Smaller equivalent depths than $\frac{7}{5} H$ (larger $\epsilon^*$) correspond to "internal" modes of the vertical structure equation. For values of static stability characteristic of the tropical troposphere, $\epsilon^* \sim 300$ for the first baroclinic mode (Geisler and Stevens 1982). This number should not be taken too seriously, however, on account of the destabilizing effect of moisture. It will be worthwhile to explore a wide range of $\epsilon^*$, noting that the stability of this region may depend on the coupling of several baroclinic modes (Chang and Lim 1988). For the present purpose we simply regard $\epsilon^*$ as a variable parameter.

Eigenfrequencies are shown in Fig. 2 for the first ten planetary zonal wavenumbers. Intermediate scale waves ($s = 6$--10) depend only weakly on $\epsilon^*$ until large values are reached, at which point the growth rates decrease rapidly. The largest growth rate for this wind profile occurs at $s = 6$, near $\epsilon^* = 140$, corresponding to an $e$-folding time of about two and one-half days.

In contrast, the low-wavenumber modes bifurcate into two branches. For high values of $\epsilon^*$, the growth rates maximize around $\epsilon^* = 500$--1000; for low values, around $\epsilon^* = 1$--20. For wave 1, a third branch exists at $\epsilon^* = 0$ (not shown), although the growth rate of this mode is extremely small and there is some sensitivity to the grid. The modes shown in Fig. 2 are robust and have much larger growth rates than the purely non-divergent mode.

Figure 2 was derived from the exact wave equation on the sphere. Results using the time-dependent method on an equatorial beta-plane are similar (see Fig. 3) except at low values of $(s, \epsilon^*)$. In that limit, the eigenfunctions are not equatorially trapped (as shown in the next subsection). Good qualitative agreement is obtained elsewhere in the figures.

Three further points should be noted:

1) The overall range of $\text{Re}(\sigma)$ extends from $\sigma_1$ to $\sigma_2$, where

$$\sigma_1 = s\sigma(\theta_1)$$ \hspace{1cm} (3.4a)

$$\sigma_2 = s\sigma(\theta_2)$$ \hspace{1cm} (3.4b)

and

$$\beta_{\text{eff}}(\theta_1) = \beta_{\text{eff}}(\theta_2) = 0$$ \hspace{1cm} (3.5)

where $\beta_{\text{eff}}$ is the mean vorticity gradient. In other words, the unstable modes have a critical latitude within the region of reversed vorticity gradient. (There is no critical latitude singularity, as the frequency is complex.) This frequency selection mechanism is consistent with Pedlosky's (1964) semicircle theorem, although no as-

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**Fig. 3.** Eigenfrequencies as in Fig. 2, but for time-dependent equatorial beta-plane model.
sumption of quasi-geostrophy has been made in our analysis.

2) The real and imaginary components of eigenfrequency are proportional to $s$ at large $e^*$ and small $s$. This dependence suggests that the latitudinal geostrophy or "longwave" approximation would be accurate for these modes (see Appendix, approximation 5). In that approximation, disturbances are nondispersive in longitude, propagating with a phase speed independent of $s$. The geopotential field necessarily has zero gradient on the equator.

3) At high $e^*$, mode trajectories in this case terminate near the neutral Kelvin wave frequency $\sigma = s/\sqrt{e^*}$. Below this terminus, the unstable eigenfrequencies lie in the "intraseasonal" range (period of about 50 days).

c. Eigenfunctions for Case I

Figure 4 shows geopotential eigenfunctions for three modes. The most unstable mode $(s, e^*) = (6, 140)$ is strongly peaked north of the unstable region, covering a relatively narrow band of latitudes. There is a large phase shift and detectable kink in amplitude as one passes through the unstable region; a small secondary amplitude maximum is then encountered near the equator. For the $(1, 2026)$ mode this secondary maximum is noticeably larger; other features are similar, including the narrow width of the eigenfunction. The third mode shown, $(1, 23)$, is very different in latitudinal scale, spanning the entire winter hemisphere. It will be recalled from Fig. 2 that this hemispheric mode belongs to a different branch of instability; it is weaker and less horizontally divergent than the others.

Evidently the confinement of the eigenfunction to a narrow band of latitudes just north of the unstable region can be attributed to one of two causes: increasing either $s$ or $e^*$ contributes to the trapping of the eigenfunction, as one might expect from the respective terms in the refractive index, viz.,

$$\frac{s^2}{1 - \mu^2} \quad \text{and} \quad e^* \Delta^*.$$

The hemispheric mode, in contrast, lacks both mechanisms of trapping, and is able to propagate throughout most of the winter westerly waveguide.

The $(1, 2026)$ mode was recalculated at higher resolution (500 grid points), and the results for $u$, $v$, and $\phi$ are shown in Fig. 5 together with the three terms of the continuity equation (A12) and their sum. Going from lower to higher resolution produces no detectable change in the wave fields and no significant change in the eigenfrequency, but the sum of continuity equation terms is somewhat smaller at higher resolution (about 1% magnitude) than at lower resolution (typically about 5%-10% magnitude). This error arises from finite differencing, probably from the differentiation of $v \cos \theta$ and calculation of $u$ near the critical latitude. Solutions of the eigenvalue problem continue to converge at even higher resolution (up to 2000 grid points).

As noted earlier, a mode such as $(1, 2026)$ would be a good candidate for the latitudinal geostrophy or "longwave" approximation, judging by the smallness of frequency and of meridional velocity. There is an approximate 7:1 ratio between the magnitudes of zonal and meridional velocity for this mode. [This is not true for the most unstable mode $(6, 140)$; in that case, $u$ and $v$ are of nearly equal magnitude.] Note, however, that the contribution from $v$ is essential to the balance of terms in the continuity equation.

d. Low-wavenumber results for Case II

Results for Case II are similar to those of Case I, except that (i) the growth rates are generally weaker; (ii) fewer planetary zonal wavenumbers are unstable ($s = 1-7$); (iii) the maximum growth rates are shifted to slightly higher $e^*$ and lower $s$. Further discussion of this case will be restricted to the $s = 1$ divergent branch of instability.

![Fig. 4. Eigenfunctions of Case I for selected modes $(s, e^*)$.](image-url)
Figure 6 shows the eigenfrequencies for this mode; beta-plane values are also shown for comparison. Growth rates maximize near $\epsilon^* = 10000$. Horizontal structures for a few eigenfunctions are shown in Fig. 7; generally speaking, their form is similar to the previous case except that the peak amplitude is located closer to the equator and the secondary maximum there is considerably larger. This secondary peak seems to reach its largest value when the growth rate itself is maximum; values of geopotential amplitude at the equator are $\sim 0.5$ for $\epsilon^*$ in the range 5000–10 000. Velocity and geopotential fields for $\epsilon^* = 5000$, shown in Fig. 8, indicate that the zonal perturbation velocity is also substantial near the equator. In fact, one might mistake this near-equatorial structure for that of an equatorial Kelvin wave, were it not for the fact that the meridional velocity, though small, makes an essential contribution to the balance of terms (cf. Fig. 5). Away from the equator, the horizontal structure is very different from a Kelvin wave; there, the primary geopotential maximum (minimum) is surrounded by anticyclonic (cyclonic) flow.

Curiously, it appears that the Kelvin wave is also destabilized by the shear profile of Case II, although the growth rates are very weak. Figure 9 shows an example, $(\epsilon, \epsilon^*) = (1, 3905)$, for which $\sigma = 0.0178 + 0.000505i$. The real part of eigenfrequency is about the same as one would expect for the Kelvin wave in no latitudinal shear. The geopotential field has a node and weak secondary maximum away from the equator. The meridional velocity, though different from zero, is about a factor of 25 less than the zonal velocity (not shown). For this mode, the primary balance of terms in the continuity equation is between geopotential tendency and zonal velocity divergence. The eigenfrequency trajectory is easily distinguished from divergent barotropic instability, the real part having the opposite dependence on $\epsilon$.

It is likely that the Kelvin wave instability, if genuine, does not belong to either the inertial or barotropic class, as the location of critical latitude is well outside the region of reversed vorticity gradient (poleward side).

4. Analysis and discussion

For an atmosphere in uniform rotation, the Laplace tidal equation governs the horizontal structure of rotational and gravitational oscillations. When latitudinal shear is added, modes of both classes may be destabilized, depending on whether certain criteria are met. In the preceding discussion, attention has been limited to flow profiles that are inertially stable and barotrop-
ically unstable. Consequently, our discussion has focused on divergent modes of barotropic instability; inertial instabilities do not arise in either of the flow profiles considered.

The restriction of inertial stability places an upper limit on the magnitude and location of reversed vorticity gradient; the gradient cannot be too large and too close to the equator without causing the vorticity itself to become negative. As it turns out, divergent barotropic instability continues to exist when the flow profile becomes inertially unstable. This appears to be the case for profiles with $U$ and $\theta$ as in Case I, but with $\theta_0$ varying from $20^\circ$ to $10^\circ$. However, over most of this range the vorticity itself becomes negative in a band of latitudes north of the equator, and inertial instabilities develop. Because barotropic and inertial instabil-

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**Fig. 6.** Eigenfrequencies of $s = 1$ mode for Case II: (a) Re$\sigma$; (b) Im$\sigma$. Solid (dashed) curves are derived from the exact spherical (beta-plane) models.

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**Fig. 7.** Eigenfunctions for selected modes of Case II.
ities coexist (Stevens and Ciesielski 1986), the behavior of eigenfrequencies is somewhat more complicated. Since the inertially unstable profiles are probably not of great importance to the troposphere, these results will not be discussed further. (On the other hand, they are relevant to the tropical mesosphere; this will be the topic of a separate investigation.)

One way to bypass the inertial instability is to make an approximation, such as latitudinal geostrophy, that eliminates these modes from the horizontal structure equation. It has already been noted that divergent barotropic instabilities at large $\epsilon^*$ and small $s$ are accurately captured by this approximation. I have elected not to show results obtained with (A33) because the $s = 1$ eigenfrequencies and horizontal structures for Cases I and II can hardly be distinguished from the exact results presented above.

The form of (A33) makes it clear that the vorticity gradient is responsible for the instability, but one can invoke a more drastic approximation than (A33). Consider the prototype equation

$$\frac{d}{d\mu} \left( \frac{1 - \mu^2}{\mu^2} \frac{d\phi}{d\mu} \right) - \frac{\phi}{\mu^2} \left( \frac{s}{\beta_{eff}^* + \epsilon^* \mu^2} \right) = 0 \quad (4.1)$$

where $\beta_{eff}^*$ is the nondimensional vorticity gradient. The underlying assumption in (4.1), besides latitudinal geostrophy, is that the nondimensional vorticity can be set equal to its planetary value $\mu$ except where differentiated—a kind of generalized beta-plane approximation (not to be confused with the quasi-geostrophic approximation). By applying the same iteration technique to (4.1) it can be shown that divergent barotropic instability exists in this prototype equation. Figure 10 shows two examples of geopotential eigenfunctions. The growth rates (not shown) are generally weaker than those reported in the previous section and exist over a somewhat narrower range of $\epsilon^*$. The form of the eigenfunction, nevertheless, is similar to that obtained with the exact equation.

One advantage of (4.1), as opposed to (A33), is that the inertially unstable region is irrelevant to the equation. In fact, the dot-dash profile in Fig. 10 was obtained for a flow profile like Case I except that $\theta_0 = 15^\circ$. For this profile, $\beta^*$ is significantly negative north of the equator. By using (4.1), we can move $\theta_0$ arbitrarily close to the equator with no effect whatever from the

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**Fig. 8.** Wave fields for eigenmode $(1, 5000)$ of Case II. Note that velocity vectors are not scaled according to the aspect ratio of the figure.

**Fig. 9.** Kelvin wave mode $(1, 3905)$ of Case II.
inertially unstable region. As before, the secondary amplitude maximum at the equator increases as the shear zone is moved closer to the equator. It also increases with $\kappa^*$ (at least up to a certain point); the eigenfunctions shown in Fig. 10 are close to the upper limit of $\kappa^*$ for which instability exists. Growth rates, on the other hand, tend to decrease as the shear zone is moved towards the equator.

In summary, results obtained with the prototype equation (4.1) further establish that the reversal of vorticity gradient is responsible for the instability, and that the qualitative behavior of eigenfrequency and horizontal structure does not depend on variations of basic state vorticity except where $\tilde{Z}^*$ is differentiated in (A33). In the low-wavenumber limit, (A33) is a suitable approximation to (A25) for the class of strongly divergent disturbances investigated here.

Equation (4.1), incidentally, also contains the weak Kelvin wave instability noted at the end of the previous section. For this mode, the growth rates tend to increase as the shear zone is moved closer to the equator.

Divergent barotropic instability exists in the quasi-geostrophic equation

$$\frac{d}{d\mu} \left( (1 - \mu^2) \frac{d\phi}{d\mu} - \phi \frac{s}{\delta} \beta^*_0 + e^* \mu^2 \right) = 0$$

although the solution variable $\phi$ would not, in this case, be interpreted as geopotential (but rather as streamfunction, say) due to latitudinal geostrophy. The unstable mode of (4.2) does not exhibit the equatorial Kelvin-like structure as before. [Note that (4.1) on the equatorial beta-plane without shear contains the neutral Kelvin wave as an exact solution.] The unbalanced Kelvin-like behavior in Case II emerges in the vicinity of the Kelvin wave trajectory, as noted above, and is not a precise coalescence. Comparing Fig. 6 with Figs. 2 and 3, it is apparent that the eigenfrequency is also slightly affected near this point. It is unclear at this time whether the eigenfunction is analogous to the so-called "Rossby–Kelvin instability" (Sakai 1989) as the existence of divergent barotropic instability does not depend on departures from quasi-balanced motion even though the structure and perhaps growth rate are significantly affected by the Kelvin wave. Conservation of pseudomomentum between the various components of the instability is essential, of course, to the structure and existence of the unstable mode. That is, it must either be of Rossby–Rossby or Rossby–Kelvin type, or some combination of the two.

5. Conclusion

It has been shown that divergent modes of barotropic instability exist in monotonic mean flow profiles having a single, narrow region of reversed vorticity gradient in tropical latitudes. In the low-wavenumber limit, these instabilities bifurcate into weakly divergent modes of hemispheric scale, and strongly divergent modes trapped about the source region, i.e., confined to tropical latitudes. Disturbances in the second category have a Kelvin-like "tail" penetrating into the deep tropics. Frequency selection is achieved by a critical latitude within the region of reversed vorticity gradient.

Weakly divergent modes of hemispheric scale are expected to be important in the construction of extratropical Rossby wavetrains, as suggested for nondivergent barotropic instability by Simmons et al. (1983). They are, no doubt, also a part of the extratropical response to tropical forcing; the response of a barotropically stable flow has been studied by Garcia and Salby (1987). It would be worthwhile, then, to generalize the results of Simmons et al. to include divergent disturbances on a zonally varying flow or, alternatively, to generalize the approach of Garcia and Salby to include zonal variations of the basic state, with local regions of reversed vorticity gradient.

Strongly divergent modes, on the other hand, have not received much attention. It is logical to expect that these modes will be as important, if not more important, than the weakly divergent hemispheric modes (at least for the ultra-long waves $s = 1$). For the mean flow profiles examined here, the location of critical latitude implies that Re(\sigma) lies in the "intrasessional" range for $s = 1$. Accuracy of the latitudinal geostrophy or "longwave" approximation implies that the next several wavenumbers propagate with nearly the same zonal phase speed as $s = 1$.

It is well known that there is considerable variance in the tropical circulation and convection on these space and time scales (Madden and Julian 1971, 1972; Parker 1973). For those authors, and many since, the Kelvin wave has served as a theoretical model for intraseasonal oscillations, although it was recognized at the outset that the first baroclinic Kelvin mode of the
tropical troposphere is not a low-frequency oscillation
(Parker 1973). Subsequent theoretical studies have
examined the effects of moist processes on Kelvin wave
propagation, and have achieved realistic near-equator-
ial structure and phase progression.

The horizontal structure of observed intraseasonal
oscillations includes a Kelvin-like structure near the
equator and coupled Rossby-like gyres away from the
equator (Knutson and Weickmann 1987; Hayashi and
Golder 1988). As noted by Hayashi and Golder, an
eastward propagating Rossby wave in a resting basic
state suggests tropical forcing [e.g., as first illustrated
by Matsumu (1966)]. It seems possible, however, that
the Rossby component is actually propagating to the
west relative to the basic state in which it is embedded
(being located 20°–30° away from the equator). This
fact does not preclude tropical forcing, but it does sug-
gest that such forcing may not be required, if the basic
state is barotropically unstable. Furthermore, the
divergent barotropic instability provides a mechanism
whereby a Rossby component is “fused” to a Kelvin-
like component. On account of the latitudinal shear,
both components propagate in their natural direction
relative to the fluid (to west and east, respectively).

Unfortunately, there is some uncertainty concerning
the value of $c^*$; if we take $c^* \sim 300$ for the first
baroclinic mode, the divergent barotropic instability has
a reasonable phase speed but insignificant equatorial
amplitude. Therefore, it may be significant that moist
processes can couple baroclinic modes together (Chang
and Lim 1988). Their study was restricted to the equa-
torial plane, but there is no reason, in principle, why
it could not be extended to include the horizontal
structure and Rossby components. At the same time
it will be necessary to introduce vertical shear and to
determine whether, in practice, an instability excited
in the upper troposphere can significantly penetrate
the lower troposphere so as to excite low-level moisture
convergence.

In conclusion, this study has shown that latitudinal
shear is dynamically important for the planetary-scale
wave modes of the tropical atmosphere, not only as
means of altering their structure, but as a source of
instability. The best-known instabilities are inertial
and barotropic instability; we have limited our discussion
to the latter although both types of instability can co-
extist (Stevens and Ciesielski 1986). A future publica-
tion by the author will examine this general case for
zonally uniform and nonuniform flows, with applica-
tion to the tropical mesosphere.

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APPENDIX

Derivation of Laplace’s Tidal Equation
with Latitudinal Shear

Perturbation equations can be derived from Holton
[1975, Eqs. (2.19)–(2.22)] assuming that the mean
zonal flow is a function of latitude only, i.e.,

$$\tilde{u} = \tilde{u}(\theta).$$  \hspace{1cm} (A1)

More generally, it is assumed that the mean flow varies
slowly in time and weight (Boyd 1978). Neglecting all
source terms, the linear perturbation equations of zonal
and meridional velocity, temperature, and continuity
can then be expressed as follows:

$$D_t u' + \left( \frac{1}{a \cos \theta} \frac{\partial}{\partial \theta} (\tilde{u} \cos \theta) - f \right) v' + \frac{1}{a \cos \theta} \frac{\partial}{\partial \lambda} \tilde{u}' = 0$$  \hspace{1cm} (A2)

$$D_t v' + \left( f + \frac{2 \tilde{u} \tan \theta}{a} \right) u' + \frac{1}{a \cos \theta} \frac{\partial}{\partial \theta} \tilde{u}' = 0$$  \hspace{1cm} (A3)

$$D_t \phi' + N^2 w' = 0$$  \hspace{1cm} (A4)

$$\frac{1}{a \cos \theta} \frac{\partial u'}{\partial \lambda} + \frac{1}{a \cos \theta} \frac{\partial}{\partial \theta} (v' \cos \theta) + \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0 w') = 0$$  \hspace{1cm} (A5)

where $u', v'$ and $w'$ are zonal, meridional, and vertical
perturbation velocity; $\phi'$ is perturbation geopotential;
$\rho_0$ is basic state density proportional to $\exp(-z/H)$; $f$
is the Coriolis parameter $2\Omega \sin \theta$; $N^2$ is the buoyancy
frequency squared; $a$ is earth radius, and the coordi-
nates $r, z, \lambda$ and $\theta$ denote time, height, longitude and
latitude, respectively. Zonal mean temperature is pro-
portional to the vertical derivative of $\phi$ which, by virtue
of (A1) and thermal wind balance, is assumed inde-
pendent of latitude; therefore, $\phi_{\text{st}}$ is neglected in (A4).
Latitudinal variations of $N^2$ are also ignored. The op-
erator

$$D_t = \frac{\partial}{\partial t} + \frac{\tilde{u}}{a \cos \theta} \frac{\partial}{\partial \lambda}$$  \hspace{1cm} (A6)

neglects the contributions from the mean meridional
circulation $(\tilde{v}, \tilde{w})$, which formally requires that the
zonal mean source terms, including heating, be small
regardless of whether they are ultimately driven by

Because the mean flow is assumed to be locally in-
dependent of height, the thermodynamic and contin-
unity equations can be combined in the usual manner
to eliminate the height dependence of perturbation
variables, yielding the equation

$$\epsilon D_t \phi' + \frac{1}{a \cos \theta} \frac{\partial u'}{\partial \lambda} + \frac{1}{a \cos \theta} \frac{\partial}{\partial \theta} (v' \cos \theta) = 0$$  \hspace{1cm} (A7)
where
\[ \epsilon = \frac{m^2 H^2 + 1/4}{N^2 H^2} \] (A8)
is the separation constant and \( m \) is the local vertical wavenumber. Further manipulation of the wave equations begins with the substitution
\[
\begin{pmatrix}
u'(\lambda, \theta, t) \\
v'(\lambda, \theta, t) \\
\phi'(\lambda, \theta, t)
\end{pmatrix} = \Re \begin{pmatrix} u(\theta) \\
v(\theta) \\
\phi(\theta)
\end{pmatrix} \exp(i \lambda \sigma - \omega t) \tag{A9}
\]
describing a single harmonic wave in longitude and time, with nondimensional zonal wavenumber \( s \) and dimensional frequency \( \omega \). It follows that
\[
\begin{align*}
-\hat{\omega} u - \bar{Z} v + \frac{s \phi}{a \cos \theta} &= 0 \quad \tag{A10} \\
+\hat{\omega} v + f_1 u + \frac{1}{a} \frac{d \phi}{d \theta} &= 0 \quad \tag{A11}
\end{align*}
\]
and
\[
\begin{align*}
-\epsilon \phi + \frac{s u}{a \cos \theta} + \frac{1}{a \cos \theta} \frac{d}{d \theta} (v \cos \theta) &= 0 \quad \tag{A12}
\end{align*}
\]
where
\[
\bar{Z} = f - \frac{1}{\cos \theta} \frac{\partial}{\partial \theta} (\bar{u} \cos \theta) \quad \tag{A13}
\]
\[
f_1 = f + \frac{2 \bar{u} \tan \theta}{a} \quad \tag{A14}
\]
and
\[
\hat{\omega} = \omega - \frac{s \bar{u}}{a \cos \theta} \quad \tag{A15}
\]
Here \( \bar{Z} \) will be recognized as the absolute vorticity of the mean flow, \( f_1 \) is a modified Coriolis parameter including the superrotation of the mean state, and \( \hat{\omega} \) is the intrinsic frequency of the wave. Note that for solid body rotation, i.e.,
\[
\bar{u} = 2 \Omega a \bar{\sigma} \cos \theta \quad \tag{A16}
\]
where \( \bar{\sigma} \) is constant in latitude, the intrinsic frequency is likewise constant, and both \( \bar{Z} \) and \( f_1 \) become equal to \( f(1 + 2 \bar{\sigma}) \). Therefore a fluid in solid body rotation can, without loss of generality, be treated as if \( \bar{u} = 0 \) by choosing the appropriate value of \( \bar{\sigma} \). Here, we are interested in those flow profiles for which the angular rotation rate \( \bar{\sigma} \) varies in latitude.

The Laplace tidal equation is obtained for the geopotential field \( \phi \) by first solving for \( u \) and \( v \) in terms of \( \phi \). In matrix form,
\[
\begin{pmatrix} -\hat{\omega} & -\bar{Z} \\ f_1 & \hat{\omega} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -k \phi \\ -d \phi / dy \end{pmatrix} \tag{A17}
\]
where
\[
k = k(\theta) = -\frac{s}{a \cos \theta} \quad \tag{A18}
\]
is the dimensional zonal wavenumber and \( y = a \theta \) is dimensional latitude. It follows that
\[
\begin{align*}
u &= \begin{pmatrix} -k \phi \\ -d \phi / dy \end{pmatrix} = \frac{-k \hat{\omega} - \bar{Z} d \phi / dy}{\Delta} \quad \tag{A19} \\
v &= \begin{pmatrix} -\hat{\omega} \\ -k \phi \end{pmatrix} = \frac{\hat{\omega} d \phi / dy + f_1 k \phi}{\Delta} \quad \tag{A20}
\end{align*}
\]
These expressions are substituted into the continuity equation (A12), and after about a page of algebra, we obtain
\[
\begin{align*}
\frac{1}{a^2 \cos \theta} \frac{d}{d \theta} \left( \frac{\cos \theta}{\Delta} \frac{d \phi}{d \theta} \right) + \frac{1}{a^2 \Delta} \frac{d \hat{\phi}}{d \theta} + f_1 \frac{d \phi}{d \theta} + \left( f_1 \frac{d \Delta}{d \theta} - \frac{\Delta}{\Delta^*} \frac{d f_1}{d \theta} \right) &= \phi \Delta \quad \tag{A22}
\end{align*}
\]
Now
\[
\frac{1}{a^2 \Delta} \frac{d \hat{\phi}}{d \theta} = -\frac{k}{a^2 \Delta} \left( \frac{\partial \bar{u}}{\partial \theta} + \bar{u} \tan \theta \right) \quad \tag{A23}
\]
since \( dk/d \theta = k \tan \theta \), and
\[
\frac{k}{a \Delta} (f_1 - \bar{Z}) = \frac{k}{a \Delta} \left( \frac{\partial \bar{u}}{\partial \theta} + \bar{u} \tan \theta \right) \quad \tag{A24}
\]
Consequently the bracketed terms multiplying \( d \phi / d \theta \) in (A22) cancel and the geopotential equation is
\[
\begin{align*}
\frac{1}{\cos \theta} \frac{d}{d y} \left( \frac{\cos \theta}{\Delta} \frac{d \phi}{d y} \right) + \frac{\phi}{\Delta} \left[ k^2 + \frac{k}{a \Delta} \left( f_1 \frac{d \Delta}{d y} - \frac{\Delta}{\Delta^*} \frac{d f_1}{d y} \right) \right] &= \epsilon \phi \quad \tag{A25}
\end{align*}
\]
A nondimensional form of (A25) is useful for some purposes. Therefore we define \( \mu = \sin \theta \), \( d \mu = \cos \theta d \theta \),
\[
f_1 = 2 \Omega a \bar{\sigma}, \quad \bar{\sigma} = \sigma - s \bar{\sigma}, \quad \Delta = (2 \Omega)^2 \Delta^*, \quad \text{and} \quad \epsilon^* = (2 \Omega)^2 \epsilon.
\]
Then
\[
\frac{d}{d \mu} \left( \frac{1 - \mu^2}{\Delta^*} \frac{d \phi}{d \mu} \right) - \frac{\phi}{\Delta^*} \left( \frac{s^2}{1 - \mu^2} \right) + \frac{s^2}{\delta \Delta^*} \left( f_1 \frac{d \Delta^*}{d \mu} - \Delta^* \frac{d f_1^*}{d \mu} \right) = \epsilon^* \phi \quad \tag{A26}
\]
Although Eqs. (A25) and (A26) are general, they have received scant attention in the literature (e.g., Tung 1979; Boyd 1982; Ahlquist 1982). More commonly, attention has focused on several special cases of interest.

1) NO LATITUINAL SHEAR

In this case, \( \tilde{\omega} = \omega, \tilde{Z} = f_1 = f \), and \( \Delta = f^2 - \omega^2 \), yielding the conventional Laplace tidal equation

\[
\frac{d}{d\mu} \left( \frac{1 - \mu^2}{\mu^2 - \sigma^2} \frac{d\phi}{d\mu} \right) - \frac{\phi}{\mu^2 - \sigma^2} \left[ \frac{s^2}{1 - \mu^2} + \frac{s}{\sigma} \left( \frac{\mu^2 + \sigma^2}{\mu^2 - \sigma^2} \right) \right] = \epsilon \phi. \tag{A27}\]

Solutions are Hough functions, with positive and negative eigenvalues \( \epsilon^* \) (Flattery 1967; Longuet-Higgins 1968).

2) EQUATORIAL BETA-PLANE

In this case, \( f = \beta y \), factor of \( \cos \theta \) are set equal to unity, and \( f_1 = f \), so that

\[
\frac{d}{dy} \left( \frac{\phi_x}{\Delta} \right) - \frac{\phi}{\Delta} \left[ k^2 + \frac{k \beta}{\omega} (y \Delta_y - \Delta) \right] = \epsilon \phi \tag{A28}\]

where \( \Delta = \beta y (\beta y - \tilde{u}_y) - \tilde{\omega}^2 \). Without shear, solutions for meridional velocity are Hermite functions (e.g., Matsuno 1966). Some effects of shear have been discussed by Boyd (1978), Boyd and Christidis (1982), and Dunkerton (1983a,b).

3) ZONALLY SYMMETRIC PERTURBATIONS

For \( s = 0 \),

\[
\frac{d}{d\mu} \left( \frac{1 - \mu^2}{\Delta} \frac{d\phi}{d\mu} \right) = \epsilon a^2 \phi. \tag{A29}\]

The zero-frequency, without shear, has been discussed by numerous authors, including Leovy (1964), Plumb (1982), Garcia (1987), and Dunkerton (1988). Analytic solutions with negative \( \epsilon \) can be derived in terms of spheroidal wave functions (Flattery 1967). Some effects of shear have been highlighted by Dunkerton (1981, 1989) and Stevens (1983); they have described, for example, the unstable positive \( \epsilon \) eigenmodes of (A29) representing equatorial inertial instability.

4) NONDIVERGENT BAROTROPIC FLOW

Unlike the preceding cases, the nondivergent limit \( \epsilon \to 0 \) is more easily approached from the original equations (A2–5) utilizing the vorticity equation with horizontal streamfunction \( \psi \) as the dependent variable:

\[
\left( \frac{\partial}{\partial \tau^*} + \tilde{\omega} \frac{\partial}{\partial \lambda} \right) \nabla^2 \psi + \frac{\partial}{\partial \lambda} \frac{\partial \tilde{Z}^*}{\partial \mu} = 0 \tag{A30}\]

where \( \tau^* = 2\Omega \mu \) and \( \tilde{Z} = 2\Omega \tilde{Z}^* \). Without shear, exact solutions are the associated Legendre functions (Rossby–Haurwitz waves). Examples of barotropic instability associated with mean shear satisfying the instability criterion have been discussed by Hartmann (1983), Dunkerton (1987), and others.

5) LATITUDINAL GEOSTROPHY

Certain solutions of the exact Laplace tidal equation (A27) can be approximated by assuming geostrophic balance in the \( y \)-momentum equation (A11); i.e.,

\[
f_1 u + \frac{d\phi}{dy} = 0. \tag{A31}\]

In particular, the Kelvin and \( n > 0 \) Rossby modes survive this approximation, which assumes the form

\[
\Delta \to f_1 \tilde{Z}. \tag{A32}\]

Simultaneously, \( k^2 \) is neglected inside the brackets of (A25). Thus,

\[
\frac{1}{\cos \theta} \frac{d}{dy} \left( \frac{\cos \theta}{\Delta} \frac{d\phi}{dy} \right) - \phi \frac{k}{\Delta \omega} \left( f_1 \frac{d\Delta}{dy} - \frac{\Delta}{\Delta \omega} \frac{d\tilde{f}_1}{dy} \right) = \epsilon \phi. \tag{A33}\]

The difference of terms involving \( f_1 \) and \( \Delta \) inside the parenthesis is proportional to the mean vorticity gradient \( d\tilde{Z}/dy \). Also, the parametric dependence on zonal wavenumber \( s \) is eliminated, since \( s \) appears only in the ratio \( \tilde{\omega}/k \). Consequently, the nondimensional eigenfrequency \( \sigma \) is proportional to \( s \). In addition to this simplification, \( \Delta \) now depends only on the basic state and is therefore real; the location of "inertial latitudes" (\( \Delta = 0 \)) is independent of the frequency. This approximation is particularly well suited to the Kelvin wave, for which both the frequency and meridional velocity are small (Boyd and Christidis 1982).

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