Nonlinear Propagation of Zonal Winds in an Atmosphere with Newtonian Cooling and Equatorial Wavedriving

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ABSTRACT

Advection of angular momentum by the mean meridional circulation is important in the quasi-biennial and semiannual oscillations of the tropical middle atmosphere. The advection is nonlinear, implying a finite horizontal or vertical displacement of angular momentum surfaces. Horizontal advection contributes to the easterly phase of the semiannual oscillation, and is sensitive to extratropical body forces. The mean meridional circulation may be thought of as a hybrid Hadley/body-force circulation driven by radiative heating and Eliassen-Palm flux convergence. Realistic steady states are obtained when a mesospheric friction layer, representing gravity wave drag, is included in the problem. This device resolves an ambiguity in the inviscid theory of the middle atmosphere Hadley circulation. Nonlinear advection is also important in the quasi-biennial oscillation; it is responsible, in part, for the strong asymmetry between east and west phases. Diabatic advection of westerly shear displaces angular momentum surfaces downward at the equator in agreement with observations. From this initial condition, it is shown that a self-propagating westerly jet is excited that differs substantially from the linear-diffusive propagation discussed by Dickinson.

These results are derived from high-resolution, two-dimensional models of the atmosphere. Realistic simulations of the quasi-biennial and stratosphere semiannual oscillations are obtained without ad hoc forcing of semiannual easterlies. It is argued, however, that a spectrum of Kelvin or gravity waves may be necessary for the westerly acceleration phase. A novel result is that the period of the quasi-biennial oscillation is increased by extratropical body forces, due to the time mean Brewer–Dobson circulation.

1. Introduction

Conservation of angular momentum is a foundational principle in dynamical meteorology. But it was not until fairly recently that this law in its exact finite-amplitude form was applied to axially symmetric circulations (Schneider 1977; Held and Hou 1980; Schneider 1983; Lindzen and Hou 1988; Dunkerton 1989a). The importance of nonlinear advection in this problem derives from a positive feedback between advection and forcing terms. An angular momentum constraint prevents the flow from attaining thermal equilibrium; hence, a thermally driven mean meridional circulation persists indefinitely. This circulation displaces angular momentum surfaces by a finite amount, so the conservation law must be satisfied exactly, without linear approximations. Remote body forces can intensify the circulation (Dunkerton 1989a). Equilibration is achieved only when the surfaces of angular momentum become parallel to mass streamlines or body forces balance the meridional transport. For the nearly inviscid, rectangular cell of Held and Hou (1980) the latitudinal gradient of angular momentum tends to zero in the upper branch, and a simple theoretical model of the Hadley circulation can be constructed. To derive this steady state, one simply assumes that the disturbed upper-level flow is an angular momentum-conserving parabola confined between two latitudes. Outside of these latitudes, the flow is in thermal equilibrium. The choice of parabola and terminal latitudes form the three unknowns of the nearly inviscid problem that are determined from the integrated equations of thermal wind balance, potential temperature conservation, and continuity of temperature. Held and Hou (1980) presented solutions for thermal equilibria symmetric about the equator. Their symmetric solutions have been generalized to solstitial conditions by Schneider (1983), Lindzen and Hou (1988), and Dunkerton (1989a). Lindzen and Hou (1988) observed that small displacements of the thermal equilibrium off the equator can magnify the winter circulation cell at the expense of the summer cell, in agreement with observations (Oort 1983). Schneider (1983) and Dunkerton (1989a) considered more extreme displacements for which the solution is essentially that of a single Hadley cell. Dunkerton noted that the strength and extent of the cell is increased by a negative body force in the winter hemisphere, and argued that this effect enhances cross-equatorial advection of angular momentum in the solstitial middle atmosphere. Such

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a force is necessary to maintain the polar night jet near its observed magnitude when the radiative forcing would accelerate the westerlies to several hundred meters per second.

In fact, this theoretical model of the middle atmosphere Hadley circulation produces realistic results only when the flow is bounded vertically at some level near the stratopause. Several experiments were described by Dunkerton (1989a) with a rigid lid placed in the upper stratosphere where the thermal forcing was tapered to zero. These simulations gave reasonable results for the stratospheric flow. But the theory predicts, in agreement with other numerical results, that as one expands the domain to include the mesosphere, the Hadley circulation becomes progressively deeper, wider, and generates stronger equatorial easterlies. The result shown in Fig. 19d of Dunkerton (1989a) demonstrates that if one assumes a thermal drive all the way to 84 km (as expected from radiative transfer), the inviscid equatorial easterlies achieved under perpetual solstice conditions exceed −280 m s⁻¹. It is doubtful, then, that air parcels materially conserve angular momentum anywhere in the upper mesosphere, including the summer hemisphere.

There is evidence that upward-propagating gravity and Kelvin waves transport momentum to mesospheric heights (Vincent and Reid 1983; Fritts 1984; Fritts and Vincent 1987; Fritts and Yuan 1989; Hirota 1978; Salby et al. 1984; Coy and Hitchman 1984; Hitchman and Leovy 1988). The momentum flux divergence is not known precisely, but appears to have the order of magnitude needed to compensate for Coriolis torques associated with the diabatic circulation. This assumption has been made in several theoretical studies (Lindzen 1981; Matsuno 1982; Dunkerton 1982a; Holton 1983). Further support is provided by general circulation models (Miyahara et al. 1986; Hamilton and Mahlman 1988). Therefore, it seems that the angular momentum problem is alleviated in the upper mesosphere (cf. Leovy 1964). It is unclear how the theory of the nonlinear Hadley circulation can accommodate a mesospheric “friction layer” needed to achieve realistic zonal winds. Nevertheless a modified nonlinear theory may be possible. It will be shown below that if one assumes a realistic distribution of negative body force in the winter westerlies, well beneath this mesospheric friction layer, the resulting horizontal advection of angular momentum (M) near the tropical stratopause generates an interior saddle point in M as observed. The magnitude of cross-equatorial flow and distortion of M-surfaces is proportional to the extratropical body force, and realistic equatorial easterlies are achieved when the polar night jet is constrained to its observed magnitude. Therefore, the conclusions obtained by Dunkerton (1989a) remain valid when a mesospheric friction layer is included, but the effective depth of the nonlinear circulation region is less than the depth of thermal forcing.

The base of the friction layer cannot be exactly represented by a rigid lid. There is vertical mass flux through this level due to the so-called “principle of downward influence” or downward “control” (Haynes et al. 1990). A positive drag on summer easterlies in the upper mesosphere induces upwelling into this layer from below, while a negative drag on winter westerlies forces subsidence toward lower levels (Andrews et al. 1987, p. 306). This vertical motion causes a departure from radiative equilibrium below the layer in which the drag is actually applied, and by mass continuity the effect decays exponentially in proportion to the number of density scale heights below the layer. Departures from radiative equilibrium are induced below the friction layer, and one can say that the applied body force exerts a downward influence in this manner. Even if the region below the friction layer is completely free of angular momentum sources, one cannot assume radiative equilibrium exactly. A second complication is that air parcels in the friction layer do not materially conserve angular momentum sufficiently well to allow any assumption to be made about the zonal velocity profile. It is difficult to make further theoretical progress with the “rectangular circulation” approximation of Held and Hou (1980). Despite this problem, it is interesting to note with hindsight that the actual steady state obtained with a numerical model has several features that could be anticipated from their theory if it were assumed that the effective nonlinear circulation depth extends to within a scale height or two of the friction layer. One of the more important predictions of the theory is that cross-equatorial advection increases when drag is applied in the winter hemisphere (Dunkerton 1989a). If the inviscid region is bounded by a rigid lid, the maximum distortion of M-surfaces occurs at the level of the lid. On the other hand, friction opposes the horizontal displacement of M-surfaces. An interior saddle point is produced when frictional body forces destroy unrealistically large values of angular momentum in the winter hemisphere.

It should not be surprising that extratropical body forces affect the tropical flow also. The principle of downward influence is valid in extratropical latitudes where the surfaces of angular momentum are nearly vertical. In the vicinity of the saddle point, there exist nearly horizontal surfaces of M that traverse the tropics before turning downward (or upward) in the summer subtropics. (An example of this pattern will be shown below.) With body forces in the winter hemisphere adjacent to a tropical region of nearly horizontal M-surfaces one might refer to a “principle of sideways influence” as a way of noting this sensitivity of the tropical flow.

One purpose of this paper is to determine steady-state regimes for the middle atmosphere Hadley circulation when the domain is extended to 84 km and a mesospheric friction layer is included (with additional drag in the winter stratosphere) to generate realistic
zonal winds. This work is a sequel to Dunkerton (1989a). As an application, it will be shown that the easterly phase of the stratopause semiannual oscillation can be generated in this configuration, with no ad hoc or in situ sources of easterly momemtum at the equator. But the magnitude and altitude of this “pocket” of easterlies is determined to a large degree by extratropical wave drag. Some previously reported simulations of the semiannual cycle failed to give a realistic oscillation because this advection was underestimated. The role of a spectrum of Kelvin and gravity waves in the westerly acceleration phase of the oscillation will also be noted.

A second purpose of the paper is to simulate—for the first time in a high-resolution model—the quasi-biennial oscillation (QBO) and seasonal cycle together. A method of modeling the QBO in two dimensions has been developed (Dunkerton 1985). Simulations of the ozone QBO have been done using a simple version of this technique in a photochemical model (Gray and Pyle 1989; Gray and Dunkerton 1989), but with little attention given to the effects of the seasonal cycle on the dynamical QBO. Some of these effects will be discussed below, including the role of time-mean upwelling in the equatorial lower stratosphere associated with the Brewer–Dobson circulation. The period of the dynamical QBO is apparently sensitive to this circulation.

Using a high-resolution version of the model in the lower stratosphere it will also be possible to elucidate (in better detail than before) how the QBO-induced mean meridional circulation affects the structure of east and west wind regimes. One of the more interesting features observed in the descending west phase is a narrow band of positive acceleration at the equator (Hamilton 1984; Dunkerton and Delisi 1985a). Very high-resolution simulations establish that this narrow, descending equatorial westerly jet can be simulated without any equatorial wave driving; it is strictly a diabatic circulation effect as hypothesized by Hamilton (1984). The mean flow evolution is substantially different from Dickinson’s (1968) “linear-diffusive” regime in which angular momentum perturbations decay as they descend. Dickinson’s regime does not conserve angular momentum. The true westerly jet intensifies its vertical shear as it descends and conserves angular momentum by contracting in latitude. The behavior is reminiscent of a decaying vortex pair. This is not to say that the west phase of the QBO does not require equatorial wave driving; the “solitary” westerly jet in practice becomes much too narrow and descends too slowly. However, the diabatic effect survives when equatorial wave driving is included. Conversely, the QBO east phase is markedly wider and descends more slowly than the west phase. This point will be demonstrated with equal and opposite wave forcings.

Theoretical background for the nonlinear Hadley circulation is briefly reviewed in section 2, and the numerical models used in this study are described in section 3. Discussion begins with the QBO model in section 4, and is extended to include the seasonal cycle and mesospheric flow in section 5, together with some simple integrations of the semiannual oscillation. Effects of the Brewer–Dobson circulation on the QBO will also be discussed in section 5.

2. Theory of mean meridional circulations

a. Governing equations

The equations of axially symmetric flow in log-pressure coordinates are

\begin{equation}
\frac{1}{\cos \theta} \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \tilde{u} \cos \theta - f \right) + \tilde{w} \tilde{u}_z = \tilde{F} \tag{2.1a}
\end{equation}

\begin{equation}
f \tilde{u} + \frac{\tilde{u}^2 \tan \theta}{a} + \tilde{\phi}_y = 0 \tag{2.1b}
\end{equation}

\begin{equation}
\tilde{\phi}_{xx} + \tilde{v} \tilde{\phi}_y + \tilde{w}(N^2 + \tilde{\phi}_{zz}) = \tilde{Q} \tag{2.1c}
\end{equation}

\begin{equation}
(\cos \theta)^{-1} \partial(\tilde{u} \cos \theta)/\partial y + \rho_0^{-1} \partial(\rho_0 \tilde{w})/\partial z = 0 \tag{2.1d}
\end{equation}

where gradient wind balance is assumed in (2.1b). The notation follows that of Holton (1975) in which \( \tilde{u}, \tilde{v}, \text{ and } \tilde{w} \) are mean zonal, meridional, and vertical velocity; \( \tilde{\phi} \) is geopotential, \( N^2 \) is static stability squared, \( \rho_0 = \rho_0 \exp(-z/H) \) is basic state density, \( f = 2\Omega \sin \theta \) is the Coriolis parameter, and \( \Omega, a \) are angular rotation and earth radius, respectively. Equations (2.1a–d) may represent either the Eulerian or transformed Eulerian mean (TEM) flow depending on the definition of source terms \( \tilde{F}, \tilde{Q} \) (Andrews and McIntyre 1976; Andrews et al. 1987). Here, it will be assumed that \( (\tilde{v}, \tilde{w}) \) represent the TEM circulation and \( \tilde{Q} \) is approximated by the Eulerian mean diabatic heating. From (2.1d) the residual streamfunction may be defined:

\begin{equation}
- \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \tilde{\Psi} = \tilde{v} \cos \theta \tag{2.2a}
\end{equation}

\begin{equation}
\partial \tilde{\Psi} / \partial z = \tilde{w} \cos \theta \tag{2.2b}
\end{equation}

which becomes the solution variable in the inversion of an elliptic equation derived from (2.1a–c), as discussed in section 3a below. It is useful to define a mass streamfunction

\begin{equation}
\tilde{\Psi}_{\text{mass}} = \rho_0 \tilde{\Psi} \tag{2.3}
\end{equation}

countoring streamlines of the mean meridional circulation:

\begin{equation}
- \partial \tilde{\Psi}_{\text{mass}} / \partial z = \rho_0 \tilde{v} \cos \theta \tag{2.4a}
\end{equation}

\begin{equation}
\partial \tilde{\Psi}_{\text{mass}} / \partial y = \rho_0 \tilde{w} \cos \theta. \tag{2.4b}
\end{equation}
Defining the absolute angular momentum per unit mass
\[ M = \bar{u} \cos \theta + \Omega a \cos^2 \theta \] (2.5)
(divided by \(a\)) it follows from (2.1a,c) that
\[ \rho_0 \cos \theta \dot{M}_1 + J(\bar{\psi}_{\text{mass}}, M) = \rho_0 \cos^2 \theta \bar{F} \] (2.6a)
\[ \rho_0 \cos \theta \dot{\theta}, + J(\bar{\psi}_{\text{mass}}, \theta) = \rho_0 \cos \theta \bar{Q} \] (2.6b)

where \( \Theta \) is potential temperature. For steady flow, streamlines are parallel to \( M \)-surfaces if \( \bar{F} = 0 \); that is, if the flow materially conserves angular momentum. It does not follow that \( \bar{\psi}_{\text{mass}} = \bar{Q} = 0 \), but rather
\[ \bar{\psi}_{\text{mass}} = \bar{\psi}_{\text{mass}}(M). \] (2.7)

Thermal equilibrium is generally impossible to attain in the tropics (Held and Hou 1980; Dunkerton 1989a).

b. Simple model of the nonlinear Hadley circulation

When \( \bar{Q} \) is represented by linear relaxation to thermal equilibrium, the conservation equation (2.6b) implies that temperature integrated over the area enclosed by a streamline is equal to the equilibrium temperature integrated over the same area:
\[ \int \int_S \rho_0 c_r(\Theta - \Theta_0) \, du \, dz = 0 \] (2.8)

where \( \mu = \sin \theta \) and \( \alpha_r \) is the thermal relaxation rate. If the streamlines or \( M \)-surfaces are approximately vertical, the area integral may be calculated over a rectangle bounded by sidewalls at \( \mu = \mu_x \), \( \mu = \mu_y \) extending from the surface to the top of the domain (or maximum altitude of thermal forcing). This approximation was made by Held and Hou (1980) and extended to the quasi-compressible case by Dunkerton (1989a). It should be recognized as an approximation to the real atmosphere in which angular momentum surfaces are not exactly vertical. Finally, it is assumed that the mean flow at the top of the domain is an angular momentum-conserving parabola to within an unknown constant. This constant and the boundaries of the circulation are to be determined as part of the solution.

Even in this simple model, solutions are difficult or impossible to obtain in closed form; an equal-area method is more useful (Held and Hou 1980). Their graphical construction illustrates basic properties of the nonlinear circulation, such as the width of the circulation cell(s) and manner in which the flow is driven away from rather than towards thermal equilibrium in the tropics. It has not been previously noted in the context of this theoretical model that extratropical friction in the winter hemisphere enhances the tropical circulation, although this fundamental result was prominent in numerical calculations. The reason may be seen in Fig. 1b if, for instance, there is a flat temperature profile in the winter hemisphere, polewards of \( \bar{u} = 0 \), in place of thermal equilibrium (as in Fig. 1a; see also Fig. 6 of Dunkerton 1989a; and the paper by Held and Hou (1980) for further discussion of the equal-area rule). The flat profile is taken to represent the effect of strong drag on the extratropical westerly flow, such as might occur during a stratospheric major warming. In this case, the area enclosed between \( \Theta \) and \( \Theta^\circ \), integrated as in (2.8), is substantially greater than before, and stronger equatorial easterlies are required.

![Graphical construction of potential temperature conservation](image-url)
(with a deeper temperature profile at the equator) to compensate for the increased area. This simple argument leads us to expect that remote body forces in the winter hemisphere will affect the steady tropical flow, as one might also expect as part of the transient response to an imposed body force.

3. Numerical models

Two numerical models were used in this study; both were derived from Dunkerton (1989a). The first model (hereafter Model I) was designed to simulate the QBO alone with very high vertical resolution. It extends from 17 to 32 km and from pole to pole on a sine latitude grid. Ninety-seven grid points were used in both directions, providing a vertical resolution of 156 m and a horizontal resolution near the equator of about 1° latitude. This vertical resolution is comparable to what has been used in one-dimensional studies of the QBO; it is almost an order of magnitude finer than Dunkerton’s (1985) two-dimensional model (1 km). Results from Model I will be discussed in section 4.

The second model (hereafter Model II) uses the same 97 × 97 array size but extends from the surface to 84 km and from pole to pole. The vertical resolution of Model II, though reduced a factor of five from Model I, is still better than Dunkerton’s (1985), and that of Gray and Pyle (1989) (3.5 km). While there is some degradation of QBO simulations relative to Model I, Model II allows the seasonal cycle to be included with an overlying mesosphere and can therefore produce a more realistic middle atmosphere response to differential heating than the model of Dunkerton (1989a). Model II has been used in section 5 to study the effects of the seasonal cycle on the quasi-biennial and semiannual oscillations.

a. Solution method

The solution method is identical to that of Dunkerton (1989a); the governing equations (2.1a–d) are reduced to an equation for the mean meridional streamfunction (2.2a,b) after replacing gradient wind balance with geostrophic balance in (2.1b) and neglecting \( \psi_{xx} \) relative to \( N^2 \) in (2.1c). The resulting equation (Eq. A1 of Dunkerton, 1989b) is of mixed elliptic/hyperbolic type when the near-equatorial flow is inertially unstable. Lacking a better parameterization of inertial instability, the technique of Dunkerton (1989a) was adopted in which the coefficients of the streamfunction equation contributing to hyperbolic behavior were reset to zero if necessary (the “parabolic” approximation). No inertial adjustment was used in the mean flow equations.

The streamfunction equation was inverted with routines from the MUDPAK library with boundary conditions impermeable to \( \psi \) everywhere. Mean velocities were obtained from (2.2a,b) and inserted into (2.1a). Upstream differencing was used for the horizontal advection of angular momentum. The time step used in Model I is 34 560 s; in Model II, 8640 s.

b. Thermal forcing and dissipation

For Model II, the seasonal cycle was driven by simple relaxation to thermal equilibrium as in Dunkerton (1989a):

\[ \bar{Q} = \alpha_T(0)(\bar{\phi}_E - \bar{\phi}_C). \]  

(3.1)

In terms of temperature \( \bar{T}^E \), the thermal equilibrium consists of a temporally constant part symmetric about the equator, and a sinusoidally varying part antisymmetric about the equator (Dunkerton 1989a); an example of \( \bar{T}^E \) at the NH solstice is shown in Fig. 2. This profile is similar to the radiative equilibrium of Wehrbein and Leovy (1982) (less global average), although the gradient near the polar night is smoothed somewhat. A token troposphere was included with \( \bar{T}^E \) similar to the observed temperature so that quasi-realistic jet streams are generated without eddy transport. (A consequence is that the tropospheric Hadley circulation is much weaker than observed.) The symmetric part of Fig. 2 was held constant in time, while the antisymmetric part varied with the annual cycle maximizing at the solstices. The length of year was taken to be 360 days with solstices at day 0, 180, 360, etc.

For Model I, \( \bar{T}^E = 0 \) as in Dunkerton (1985). All previous QBO studies with the exception of Gray and Pyle (1989) and Gray and Dunkerton (1989) have made this assumption. It turns out to be quite critical to the QBO dynamics as shown in section 5 below.

The Newtonian cooling coefficient for the zonal mean state is denoted \( \alpha_T(0) \) (where “0” signifies in-

![FIG. 2. Thermal equilibrium temperature of Model II at the NH solstice (less global average). Contour interval: 10°K.](image)
finite vertical wavelength or radiation-to-space) and shown in Fig. 3. Values are similar to those of Fels (1982) in the middle atmosphere (except near 84 km, where tapered to zero). A uniformly rapid relaxation (time scale of about 7 d) was used in the troposphere.

Thermal relaxation was also used to damp vertically propagating equatorial waves (see next subsection). The coefficient depends on vertical wavenumber $m$ with a square-root dependence approximating that of Fels (1982). An example of this enhanced radiative damping is shown in Fig. 3 for a vertical wavelength of $2\pi$ km. The scale dependence is an important effect for both semiannual (Dunkerton 1979) and quasi-biennial (Hamilton 1981) oscillations.

Model II used the $\alpha_T$ profiles shown in Fig. 3; a linear fit to these profiles was used in Model I. For a few experiments with Model I, the $\alpha_T(0)$ profile was increased by a factor of four to mimic the radiative damping of shallow QBO mean temperature anomalies.

Model II also included a height-dependent Rayleigh friction coefficient $\alpha_M$ (Schoebel and Strobel 1978; Holton and Wehrbein 1980) shown in Fig. 3. The time scale asymptotes to two days in the upper mesosphere. A small friction was used near the tropopause to damp QBO regimes descending into the tropical troposphere.

While not necessary for the QBO switching mechanism (Plumb 1977), this friction parameterizes the effects of strong Hadley upwelling, cumulus transport, and eddy mixing that would have a similar effect in reality. No Rayleigh friction of any kind was used in Model I. Instead, $\vec{u} = 0$ at 17 km (Holton and Lindzen 1972). In Model II, the surface boundary condition utilized a nonzero drag coefficient as in Held and Hou (1980) and Dunkerton (1989a). At the upper boundary of both models $\vec{u}_z = 0$. Finally, a small horizontal ($K_{xx}$) and vertical ($\nu$) diffusion was used; $\nu = 0.3$ and 1.0 m$^2$ s$^{-1}$ in Models I and II, respectively, and unless noted otherwise, $K_{yy} = 5.0 \times 10^3$ m$^2$ s$^{-1}$ in both models.

c. Equatorial waves

Following Dunkerton (1985), the effect of equatorial waves on the transformed Eulerian mean flow was parameterized as a source term in the momentum equation (2.1a). In this method, the problem is split into two parts. First, the latitudinally averaged body force is calculated from the wave action equation (integrated across the equatorial waveguide) using formulae similar to those of Holton and Lindzen (1972). Second, the eigenvalues and horizontal structure of wave fields are approximated with analytic expressions. Latitudinal shear is neglected for the Kelvin wave; the equatorial zonal mean wind is primarily important for this wave. For the Rossby-gravity wave, the gamma-plane approximation is used (Boyd 1978). The version of the gamma-plane routine used here takes into account mean flow curvature about the equator but neglects cross-equatorial shear (a small effect in the lower stratosphere). The latitudinally averaged body force is then distributed across the waveguide with an analytic function derived from Andrews and McIntyre (1976). Details of this method are given in Dunkerton (1985).

The implementation here was very similar, except that vertical diffusive damping was omitted; instead, a small mechanical damping of the Rossby-gravity wave was assumed such that the ratio of mechanical to thermal damping equals 0.2 at all times.

Vertical propagation of the waves is governed by

$$B(z) = B(17) \exp \left[ \frac{z}{H} - P(z) \right]$$  \hspace{1cm} (3.2)

where $B$ is the vertical component of Eliassen–Palm flux (latitudinally averaged), $z$ is in kilometers, $H$ is the density scale height and

$$P(z) = \int_{17}^{z} D(z') dz'$$  \hspace{1cm} (3.3)

where $D(z)$ is the dissipation rate (m$^{-1}$). Typically $D$ varies like (damping coefficient) $\div$ (vertical group velocity) depending on the equipartition law (Dunkerton 1985). In a quasi-compressible atmosphere this quantity determines whether the waves are growing [$D(z)H < 1$] or decaying [$D(z)H > 1$] in height (Dunkerton 1985).
1979, 1982a). Prior to modeling with the wave parameterization it is helpful to understand the transmission of waves in the assumed Newtonian cooling profile. In this regard, the traditional equatorial Kelvin and Rossby-gravity waves (Wallace and Kousky 1968; Yanai and Maruyama 1966) are well suited for the QBO. However, they readily decay in the upper stratosphere (Hamilton 1981) and make only a small contribution at the stratosopause. Due to the enhanced scale-dependent damping, the low-wavenumber Hirota (1978) Kelvin wave also experiences strong dissipation in the upper stratosphere (as anticipated by Dunkerton 1979). Thus, it is likely that higher wavenumbers contribute to the stratosopause semiannual oscillation (Hitchman and Leovy 1988; Hamilton and Mahlman 1988). This point will be explored further in section 5. In section 4, the likelihood that other waves of smaller phase speed and/or higher wavenumber contribute to the quasi-biennial oscillation is noted also.

4. Symmetric QBO simulations

In this section, Model I will be used to investigate the dynamical QBO without seasonal cycle or Brewer-Dobson circulation (as in Dunkerton (1985)) but with higher resolution. Integrals of the form (3.3) can then be evaluated with vertical resolution comparable to earlier one-dimensional models. The discussion begins with the case of a diabatically driven “solitary” westerly jet.

a. Self-propagation of equatorial shear

The linear evolution of zonal winds with thermal relaxation as Newtonian cooling was examined by Dickinson (1968). This work has resurfaced—after being buried in the rubble of QBO theories—for its insights on the “principle of downward influence” in extratropical steady flow (Haynes et al. 1990). Dickinson’s comments on nonsteady tropical flow are germane to the discussion here.2 It was suggested that zonal winds induced by an oscillating momentum source in the equatorial middle stratosphere would propagate downward in a diffusive manner. This mechanism is explained if one recalls that by thermal wind balance, westerly vertical shear at the equator is warm (and vice versa), requiring subsidence to maintain the temperature anomaly against radiative relaxation (Reed 1964; Wallace 1967). Outflow beneath westerly shear induces a westerly acceleration due to the Coriolis force, and downward “propagation” is possible. To see this in mathematical terms, consider the linear mean flow equations on an equatorial beta-plane:

\[ \bar{u}_t - \beta y \bar{v} = 0 \]  
(4.1a)

\[ \beta y \bar{u} + \bar{\phi}_y = 0 \]  
(4.1b)

\[ \bar{\phi}_{yy} + \frac{N^2}{J} \bar{w} = -\alpha_T \bar{\phi}_z \]  
(4.1c)

\[ \bar{v}_y + \rho_0^{-1} (\rho_0 \bar{w})_z = 0. \]  
(4.1d)

From (4.1a) and (2.2a),

\[ \bar{u}_t = -\left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \beta y \bar{v} \approx \left( \frac{\partial}{\partial z} - \frac{1}{H} \right) \beta y L^2 \bar{\psi}_{yy} \]  
(4.2)

if a local anomaly is assumed, i.e., \( \bar{\psi} \approx -L^2 \bar{\psi}_{yy} \). Combining (4.2) with (4.1b,c) and using (2.2b) it may be shown that if the thermal damping dominates the temperature tendency in (4.1c), then

\[ \bar{u}_t \approx \alpha_T \left( \frac{\beta y L}{N^2} \right)^2 \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 \frac{\partial \bar{u}}{\partial z} \right). \]  
(4.3)

The Coriolis effect combined with Newtonian cooling “diffuses” the zonal flow vertically (Plumb and Bell 1982) with an effective diffusivity \( \alpha_T (\beta y L/N)^2 \). Downward propagation of zonal winds would presumably occur below a pulsating momentum source in the middle stratosphere (Dickinson 1968) such as that suggested by observations of horizontal eddy momentum flux (Wallace and Newell 1966).

Unfortunately, this argument (aside from being irrelevant near the equator) neglects to account for quasi-compressible effects; the signal decays rapidly below the source level (cf. Wallace and Holton 1968). In fact, as one can readily see from (4.3) the diffusive adjustment to a steady state proceeds mainly upwards not downwards. In the linear model, then, nothing actually happens on the equator, and the off-equatorial diffusion is, for the most part, in the wrong direction. More importantly, angular momentum is not conserved in a manner consistent with the exact conservation law.

It is interesting to reconsider the nonlinear evolution from a state of initial vertical shear at the equator. Beginning from westerly shear, with a warm temperature anomaly and diabatic subsidence, the shear zone is bent downward at the equator. The temperature anomaly descends, so the vertical velocity maximum descends, carrying the shear zone with it. The temperature anomaly descends further, along with the vertical velocity maximum, and so on. Because subsidence is largest at the center of the shear zone, vertical shear tends to steepen below the point of maximum shear. To see this, consider the nonlinear equations of symmetric flow at the equator:

\[ \bar{u}_t + \bar{w} \bar{u}_z = 0 \]  
(4.4)

2 The title is adapted from Dickinson (1968) to emphasize the point that spontaneous “propagation” of zonal winds occurs as a result of angular momentum conservation in a diabatic atmosphere. This is most apparent in the descending westerly shear zone. With less precision one may also refer to spontaneous horizontal propagation of angular momentum in the Hadley circulation (due to a feedback effect noted in the Introduction). In both cases it is the advection of angular momentum that is meant, not merely the propagation as of waves.
together with (4.1b–d). Once again neglecting the temperature tendency and assuming a localized jet,

$$\bar{u}_i \approx \alpha T \beta L^2 \bar{u}_z^2 / N^2$$  \hspace{1cm} (4.5)

or in terms of the shear zone descent rate,

$$(dz/dt)_d \approx -\alpha_T \beta L^2 \bar{u}_c / N^2.$$  \hspace{1cm} (4.6)

The descent rate is proportional to the shear, which has the interesting implication that within a region of positive shear ($\bar{u}_z > 0$) increasing with height ($\bar{u}_z > 0$) any two characteristic values of $\bar{u}$ will converge—and eventually overlap—as time proceeds. In practice, $L$ decreases with time in order to conserve angular momentum. For simplicity one might assume that $L$ decays exponentially as the center of the shear zone is displaced downwards from its initial position. Descent is slowed as the westerly jet contracts in latitude, weakening the temperature perturbation and vertical velocity. But this decay does not seem to prevent "shock" formation as a simple numerical example demonstrates.

Figure 4a shows an initial condition applied in Model I, with westerly shear centered on the equator at 28 km. The initial streamfunction, like a vortex pair, induces downward motion at the equator and upwelling in the subtropics (Fig. 4b). For this integration only, the Newtonian cooling coefficient $\alpha_T(0)$ was set equal to 0.13 $d^{-1}$ at all levels, and $\nu = K_{yy} = 0$. The large value of $\alpha_T(0)$ provides an upper limit on what can realistically be expected from diabatic descent. After 360 days the flow has evolved to the configuration in Fig. 5. The shear zone, now dramatically narrower, has descended to near 24 km and the streamfunction in comparison to Fig. 4b is much weaker. Away from the equator the Coriolis effect has produced easterlies at upper levels and a weak westerly tendency below. The asymptotic equatorial flow appears to be singular, compared to the rest of the flow, as it evolves to a very steep shear. The linear-diffusive model also produces a narrow jet, but without steep vertical shear or equatorial descent.
of $\vec{u}$ at 60-d intervals (up to 720 days). The decreasing rate of descent is apparent, and the total time required to achieve this downward displacement of about 7 km (2 years) is far too long to explain the descent of QBO westerlies (which typically descend at a rate of 1 km per month). When more realistic $\alpha_T(0)$ are used (as in Fig. 3), the profile evolves similarly but requires even longer to descend (over 4 years). In any case, the westerly jet is very narrow, within 5° of the equator and fairly insensitive to initial conditions. It may also be barotropically unstable.

This example highlights important differences between the linear and nonlinear evolution. Diabatic descent is not unrealistic due to shear zone decay (as alleged against Dickinson 1968). The nonlinear flow conserves angular momentum, preserves the initial values of $\vec{u}$, and actually steepens the initial shear at the equator. It is the contraction of the jet and long time required for descent that are unrealistic. One may conclude from this example that, although advection by itself cannot adequately explain the west phase of the QBO, the effect of diabatic subsidence might be important for the near-equatorial flow.

It is noted incidentally, the evolution from initial easterly shear is completely different; there is upwelling at the equator instead of subsidence. Outflow above the shear zone maintains the temperature perturbation against damping. It will become apparent in the next subsection that this basic asymmetry due to nonlinear advection has important consequences for the evolution of equatorial shear in connection with the QBO.

b. Kelvin and “anti-Kelvin” wavemdriving

The Kelvin wave parameterization uses equatorial zonal wind to calculate intrinsic phase speed for the integral (3.3) and other formulae. The latitudinal structure of body force is a Gaussian (Lindzen 1971; Andrews and McIntyre 1976) with scale factor

$$y_0 = \left[ \frac{(c-\vec{u})/\beta}{\beta} \right]^{1/2}. \quad (4.7)$$

The Gaussian contracts as the critical level $\vec{u} = c$ is approached. If no other body forces are present, the
The net change in mean zonal wind is just the sum of many Gaussian increments, each with a progressively smaller latitudinal scale as shown in Fig. 7. It appears from this calculation that the effective latitudinal scale of the asymptotic westerly jet is about two-thirds the initial scale of the wave. For the traditional Kelvin wave ($c \approx 25$ m s$^{-1}$) in QBO easterlies ($\bar{u} \approx -30$ m s$^{-1}$), the initial scale is about 14° latitude, and the e-folding scale of the final jet is therefore about 9° (Dunkerton 1985). This analytic example neglects mean circulation advection, and provides a useful benchmark against which diabatic effects can be measured.

It will be instructive to consider equal and opposite wave forcings. This can be done by supposing that there exists an “anti-Kelvin” wave with properties exactly like those of the true Kelvin wave but having a negative phase speed and momentum flux, therefore exerting an easterly body force on the mean flow. Of course no such waves exist on the normal equatorial beta-plane; latitudinal geostrophy requires positive intrinsic phase speed, but the parameterization scheme does not care.

Beginning with the Kelvin wave case, Model 1 was integrated from an initial state similar to Fig. 4a (with slightly deeper shear zone) and including a single Kelvin wave with phase speed $c = 30$ m s$^{-1}$, zonal wavenumber $s = 1$, and $B(17) = 4 \times 10^{-3}$ m$^2$ s$^{-2}$. For this simulation $K_{yy} = 0$, and $\alpha_T(0)$ was increased a factor of four relative to Fig. 3 to mimic the effect of a shallow mean temperature perturbation (Fels 1982). The resulting mean zonal flow and meridional streamfunction after 94 days is shown in Figs. 8a,b. Comparison to Figs. 5a,b indicates that the westerly jet is wider due to Kelvin wavemdriving, and the streamfunction is considerably stronger (for the same reason). Some contraction of the jet relative to the analytic profile (Fig. 7) is seen. The most obvious difference is the descent rate, which in Fig. 8 required only 94 days to reach 23.5 km, compared to the much slower diabatic descent of Fig. 5. Kelvin wavemdriving provides a source of westerly angular momentum that maintains the latitudinal scale of the jet and associated warm temperature anomaly. Steepened vertical shear is a consequence of wavemdriving due to “underdamped” ($DH < 1$) waves (e.g., Dunkerton 1982a). Diabatic advection bends the shear downwards at the equator (Fig. 8a). The two effects reinforce one another, but wavemdriving is essential for a realistic westerly jet.

Now consider the “anti-Kelvin” case, designed as a mirror-image experiment. Phase speed and momentum flux of the anti-Kelvin wave were assumed equal and opposite to the values above, and the initial state contained easterly winds on top of westerly. For the same Newtonian cooling rate, 360 days were required for the easterly shear zone to reach 23.5 km, as shown in
Fig. 8. Mean zonal wind (a) and streamfunction \( \psi \) (b) for the Kelvin wave experiment, at \( t = 94 \) d. Units: m s\(^{-1}\) and m\(^2\) s\(^{-1}\). Westerlies are shaded in (a).

Fig. 9. The easterly jet is much broader than the westerly jet of Fig. 8a; it is relatively flat at the center and flanked by minor westerlies. This westerly flow cannot be attributed directly to wave driving, which is everywhere negative for the anti-Kelvin wave. It is a consequence of angular momentum transport due to poleward outflow above the shear zone. The effect of upwelling at the equator is clearly visible. A wider jet implies a larger temperature perturbation which, in turn, induces a slightly stronger circulation cell in Fig. 9b relative to Fig. 8b. Once again the linear model is inadequate as the angular momentum gradient is significantly reduced in the outflow layer above the shear zone.

Differences between the Kelvin and anti-Kelvin experiments are due entirely to diabatic advection. It is worth noting that values of mean vertical shear and temperature are not unrealistic when compared to the actual QBO. Estimated diabatic descent in the QBO west phase (from 4.6) is comparable to the estimated wave-induced shear zone descent

\[
(dz/dt)_{wave} \approx -B(z)/\Delta \tilde{u} \tag{4.8}
\]

where \( \Delta \tilde{u} \) is the net zonal wind change between QBO phases—even when slow Newtonian cooling rates are adopted. Diabatic advection is therefore important in the QBO (Lindzen and Holton 1968; Plumb and Bell 1982), and the observed narrow westerly acceleration phase (Hamilton 1984; Dunkerton and Delisi 1985a).

Equations (4.6) and (4.8) may crudely delimit "diabatic" and "wave-induced" westerly descent, but it should be kept in mind that the two effects interact to reinforce one another, as noted above. (In the east phase, they are opposite in sign.) There does not seem to be any simple analytical method to estimate the diabatic contribution a priori since it depends on the magnitude and latitudinal extent of vertical shear induced by the wave driving. Nevertheless it is interesting to note that by allowing the shear zone to bend downward at the equator, the temperature perturbation is nonsingular even when the wind profile contains a discontinuous "shock" [as in Dunkerton (1982b) and Coy (1983), for example]. Conceivably, one could develop a two-dimensional theory of transient, conservative wave, mean-flow interaction with diabatic advection.
c. Quasi-biennial oscillations

Diabatic advection is not the only cause of asymmetry between QBO east and west phase. The equatorial wave forcings differ in amplitude and latitudinal structure (Andrews and McIntyre 1976; Dunkerton 1985). It is pertinent to ask what magnitudes of forcing are observed, and actually needed in the QBO. Classic observations of equatorial wave parameters are summarized in Andrews et al. (1987, p. 211) and earlier references. From this list one may infer a Kelvin wave Eliassen–Palm flux of about $4-6 \times 10^{-3} \text{ m}^2 \text{ s}^{-2}$. The Rossby-gravity wave forcing is substantially less, perhaps as small as $-1 \times 10^{-3} \text{ m}^2 \text{ s}^{-2}$. If the QBO is driven entirely from below as suggested by Lindzen and Holton (1968) and Holton and Lindzen (1972), then the Eliassen–Palm flux necessary for the west phase is $6 \times 10^{-3} \text{ m}^2 \text{ s}^{-2}$; about two-thirds this value is required for the east phase. Thus, the observed wave forcings are marginal in comparison to those actually needed. One might argue that wave amplitudes have been underestimated and this could easily account for a quadratic flux deficit. In fact, the present calculation of the necessary flux underestimates what is really needed, because mean circulation effects have not been included, e.g., the Brewer–Dobson circulation. When this circulation is taken into account, the wave forcings must be increased, as shown in the next section. On the other hand, if the background circulation is neg-
ligible or absent [as in Holton and Lindzen (1972) and Dunkerton (1985), for example], one expects a realistic QBO to be generated by forcings close to those observed. This section concludes by displaying a few symmetric QBO simulations obtained with Model I.

Figures 10a–c illustrate several integrations with $v = 0.3 \text{ m}^2 \text{s}^{-1}$ and $\alpha_T(0)$ as in Fig. 3. In Fig. 10a, $K_{yy} = 0$, while in Figs. 10b,c $K_{yy} = 5 \times 10^3 \text{ m}^2 \text{s}^{-1}$. The only reason for including nonzero $K_{yy}$ is to expand the westerly jet (Dunkerton 1985), but the diffusivity cannot be any larger without damping the narrow westerly “nose.” Much larger $K_{yy}$ are generally “toxic” to the QBO as a whole, severely attenuating the amplitude and (in some cases, at least) decreasing the oscillation period. The model QBO cannot tolerate large values of $K_{yy}$ near the equator.4

Wave parameters in Fig. 10 are

$$c_{KQ} = 25 \text{ m} \text{s}^{-1}$$

$$s_{KQ} = 2$$

$$B_{KQ}(17) = \begin{cases} 4.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2} & \text{(Fig. 10a)} \\ 4.5 \times 10^{-3} \text{ m}^2 \text{s}^{-2} & \text{(Figs. 10b,c)} \end{cases}$$

(4.9)

for the basic Kelvin wave ($KQ$), and

$$c_{RG} = -35 \text{ m} \text{s}^{-1}$$

$$s_{RG} = 4$$

$$B_{RG}(17) = -2.5 \times 10^{-3} \text{ m}^2 \text{s}^{-2}$$

(4.10)

for the Rossby-gravity wave ($RG$). In Figs. 10a,b a second Kelvin wave ($KT$) was added:

$$c_{KT} = 15 \text{ m} \text{s}^{-1}$$

$$s_{KT} = 1$$

$$B_{KT} = 4.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2}$$

(4.11)

representing the Parker (1973) Kelvin wave. This wave is probably related to the tropical intraseasonal oscillation (TIO) discovered by Madden and Julian (1971, 1972), but has a slightly shorter period (near 35 d, compared to 40–50 d for the TIO). It is likely that a continuous frequency spectrum exists in the intraseasonal range, merging with the Wallace–Kousky (1968) $s = 1$ Kelvin wave (Salby and Garcia 1987, Fig. 3; R. Madden personal communication, 1989). The wave parameters (4.11) satisfy $DH < 1$ in the QBO east phase, so this component can contribute to the westerly acceleration phase if present.

Only slight differences are seen in Figs. 10a–c. The small horizontal diffusion rapidly damps the westerly jet maximum but has no other significant effect at the equator. The Parker Kelvin wave maintains rather constant westerly descent and produces a slight westerly bias at the lowest levels. In the “classic” QBO model (Fig. 10c), shear zones experience progressively slower descent as they reach lower levels, where the basic density is larger and a longer time is required to achieve the same change in zonal wind. It is noteworthy that QBO observations indicate remarkably constant westerly descent (Naujokat 1986). This behavior is plausible if responsibility for the west phase is shared among a spectrum of waves.

The time–height structure of these simulations agrees well with observations. The latitudinal structure is less perfect; two-dimensional model oscillations are generally narrow (Dunkerton 1985), although the defect is not terribly significant. With small nonzero $K_{yy}$, absolute westerlies extend to 8° latitude, close to what one infers from the deseasonalized QBO with time mean included (Dunkerton and Delisi 1985a). The latitudinal structure during phase transitions at 23.5 km, corresponding to Fig. 10b, is shown in Figs. 11a–d. Strong asymmetry between east and west phase is obvious. The Rossby-gravity wave effect, as parameterized here, is quite similar to that of the “anti-Kelvin” wave. Once again the temperature perturbation and diabatic vertical velocity are stronger in the easterly acceleration phase, although in this case the vertical shear is actually weaker (due to weaker forcing and mechanical damping). The temperature perturbation is sensitive to latitudinal scale $L$.

Simulations with enhanced $\alpha_T(0)$ characteristically have stronger asymmetry and a narrower westerly jet, but with a more pointed westerly “nose.” These simulations might tolerate slightly larger horizontal diffusivity; otherwise, it is impossible to generate westerly jets with realistic width and a narrow initial acceleration. This is one of the unfortunate consequences of angular momentum conservation.5

In summary, the two-dimensional model without seasonal cycle or time mean Brewer–Dobson circulation generates a fairly realistic QBO with values of equatorial wave forcing close to those observed

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4 Observations of volcanic aerosol transport give conflicting estimates of the horizontal “diffusivity” of the equatorial lower stratosphere depending on the latitude and time of the eruption. Agung (1963) and Ruiz (1985) eruptions, for example, suggest tropical confinement; but this is not always the case, as in El Chichón (1982).

5 Effects other than diabatic advection seem less likely to explain the nose. Convective or dynamical ($KH$) wavebreaking might begin at the equator (the latitude of greatest Kelvin wave amplitude), but will be difficult to achieve at such large intrinsic phase speeds. Local instability would more likely result from a nonlinear cascade due to triad interactions; this mechanism deserves to be examined. Absorption of a wide spectrum of internal gravity waves will also contribute to equatorial accelerations. On the other hand, little is gained by contracting the latitudinal scale with slower phase speed equatorially trapped waves as in (4.7) because the dissipation rate increases rapidly as the intrinsic phase speed is reduced.
(Dunkerton 1985). The Parker (1973) Kelvin wave is not necessary to the QBO, but more realistic westerly descent can be achieved with a spectrum of waves.

5. Seasonal cycle effects: Hadley circulation, QBO, and SAO

Model II, as described in section 3, includes a non-zero thermal equilibrium profile shown in Fig. 2. For seasonal integrations, the symmetric part of this profile is held constant in time while the antisymmetric part varies annually. The seasonal cycle generates a semi-annual oscillation (SAO) in the tropical upper stratosphere and mesosphere. It will be interesting to determine how the QBO affects the SAO, since model simulations of the SAO without a QBO are unrealistic.

Before examining these oscillations in Model II, the
steady-state problem posed in section 2 is now considered.

a. Nonlinear Hadley circulation

Under perpetual solstice conditions, Model II (with thermal and mechanical relaxation specified as in Figs. 2, 3) generates a steady state similar to Dunkerton's (1989a) inviscid model in the stratosphere with a strong polar night jet approaching 180 m s$^{-1}$ (not shown). However, there are significant differences at the stratopause and above. The streamfunction is not artificially confined to the stratosphere, and the resulting cross-equatorial advection of angular momentum at the stratopause is weak. In the mesosphere, the strong background Rayleigh friction (Fig. 3) suppresses the effect of horizontal advection, and both summer and winter jet streams decay with height.

It is more realistic to incorporate additional drag in the winter stratosphere. Dunkerton (1989a) introduced "one-sided" friction acting only on strong westerlies (intended to represent drag due to planetary Rossby waves and other quasi-stationary waves propagating on a westerly current):

$$\vec{u}_i + \cdots = \alpha_m (\vec{u}_m - \vec{u}) + \cdots, \text{ if } \vec{u} > \vec{u}_m. \quad (5.1)$$

Note that $\alpha_m$ is distinct from the background $\alpha_M$ of Fig. 3. This device provides a smooth drag on the polar
night jet, but does not directly affect the tropics or summer middle atmosphere at all if $\bar{u}_m$ is chosen to exceed the largest positive values of $\bar{u}$ in this region. Nevertheless, an indirect effect is likely, as argued in section 2.

The steady state with $\alpha_m = (5 \, \text{d})^{-1}$ and $\bar{u}_m = 30 \, \text{m s}^{-1}$ is shown in Figs. 12a–d, at $t = 360 \, \text{d}$. More precisely this is an “almost” steady state because very weak tendencies are still present in the lower stratosphere. In contrast, the upper stratosphere and mesosphere equilibrate within several weeks. Figure 12a reveals a polar night jet of realistic amplitude, but without the observed equatorward tilt (e.g., Barnett and Corney 1985). In reality there is a polar mesospheric “surf zone” that disrupts the westerly flow (Dunkerton and Delisi 1985b). This feature is of secondary importance here. Note the strong penetration of summer easterlies near the equatorial stratopause. The altitude of this feature (7–8 scale heights) agrees well with the observed “pocket” of easterlies in December at the beginning of the SAO east phase. Later in winter the observed easterly maximum descends as equatorial westerlies encroach from above; there is also some easterly acceleration in the equatorial middle stratosphere coinciding with polar warmings (Delisi and Dunkerton 1988). Of course, this behavior cannot be simulated in the perpetual solstice experiment of Fig. 12.

The net torque per unit mass, $D_e \cos \theta$, is shown in Fig. 12b. In addition to background friction in the mesosphere there is the effect of one-sided friction in the
winter upper stratosphere with magnitude 1–10 m s\(^{-1}\)/d, and enhanced positive drag near the tropical stratosphere. It is interesting to note that the maximum heating in the summer hemisphere, indicated by a bold line and dashed contours, closely parallels the latitude of maximum positive drag. In the summer stratosphere, the largest departure from radiative equilibrium is found in the subtropics (Dunkerton 1989a, and references).

The angular momentum distribution (unweighted by density) is shown in Fig. 12c. This field has been “punctured” by horizontal advection near the stratosphere. Due to background friction, the saddle point does not coincide with the level of maximum meridional velocity, which is noticeably higher in the middle mesosphere. The magnitudes of heating, meridional, and vertical velocity agree well with those of more complex models, indicating that our simple approach is adequate to explain this basic feature of the angular momentum distribution. Examples of observed \( M \) have been given in Haynes et al. (1990).

Figure 12d shows the mass streamfunction \( \tilde{\psi}_{\text{mass}} \),
contoured quasi-logarithmically to emphasize all levels more or less equally. Mass streamlines are parallel to $M$-surfaces in the inviscid steady state; this is very nearly the case in the summer stratosphere, tropical stratosphere, and tropopause (except at the surface). Interestingly the mass streamlines deflect slightly polewards near 17 km where a small background friction was assumed—for reasons explained in section 3—even though its effect on the zonal wind is practically invisible.

Figure 12d may be broadly summarized in terms of three streamlines, shown schematically in Fig. 13. Streamline ‘$T$’ recirculates through the tropical troposphere as in Lindzen and Hou (1988). Streamline ‘$M$’ conserves angular momentum along its vertical path in the summer and lower winter stratosphere, but is strongly affected by background friction while moving horizontally in the mesosphere (where it initially gains angular momentum), and by one-sided friction while descending polewards in the upper winter stratosphere (where it loses angular momentum). Streamline ‘$S$’ is a hybrid of the two, rising conservatively in the summer subtropics and continuing to conserve angular momentum while crossing the tropics near the saddle point. This trajectory does not recirculate immediately, but proceeds straightway into the zone of one-sided friction, crossing $M$-surfaces as it begins its descent in the winter hemisphere.

Structure of the mesospheric flow is determined, to a large degree, by frictional drag that opposes meridional transport of angular momentum due to the thermal forcing. However, the stratospheric flow is insensitive to upper mesospheric drag—as one expects from downward influence in a quasi-compressible atmosphere. [It is for this reason that Dunkerton (1989a) obtained realistic stratospheric flow.] Figure 14 demonstrates this by comparing zonal wind profiles at the equator for several simulations in addition to the control experiment of Fig. 12. Solid curves in Fig. 14 were obtained by lowering the height of the upper boundary from 84 to 63 km in 7-km intervals. Truncation of the domain artificially enhances cross-equatorial flow only in the immediate vicinity of the boundary. Dashed curves in Fig. 14 were obtained by raising and lowering the centerpoint of the background Rayleigh friction profile ($\alpha_M$ of Fig. 3) by 7 km. Equatorial easterlies increase with height, approximately exponentially, as the friction layer is raised. The stratospheric flow, on the other hand, is affected only when the friction layer is brought down to within about a scale height of the stratopause. Similar results are obtained at other latitudes (not shown).

In reality there is a tradeoff between the exponential decay of the signal due to downward influence in a quasi-compressible atmosphere, and the rather dramatic increase with height of the thermal forcing and momentum sinks. Haynes et al. (1990) have argued that in order to estimate downwelling near 18 km to within an accuracy of 20% it is necessary to take into account body forces as high as the lower mesosphere (especially in the southern winter).
How relevant is the steady-state solution in the middle atmosphere? In agreement with previous results it was found that the flow above the middle stratosphere reaches to within about 90% of its asymptotic value in 1–2 months, while the lower stratospheric flow requires longer to equilibrate. Figure 15 displays equatorial zonal wind at 4–9 scale heights as a function of time in a perpetual solstice experiment done in three stages. In the first stage, the flow was spun up from rest without any one-sided friction. As noted above, this flow has rather weak equatorial easterlies (and a very strong polar night jet). In the second stage, the one-sided friction was added as in (5.1) with parameters as in Fig. 12. The asymptotic state at \( t = 6 \) yr is similar to that of Fig. 12. Dramatic easterly accelerations are observed initially at 6 and 7 scale heights. In the third stage the one-sided friction was removed (as in stage 1). The final state at \( t = 9 \) yr is identical to \( t = 3 \) yr. However, the relaxation time is substantially longer than in the first two stages (due to background friction). The solution appears to be unique (a consequence of ellipticity), but the rate of approach to the steady state obviously depends on the dominant mechanism. It may be inferred from this simple experiment, as a practical point, that equatorial accelerations occurring in response to major warmings are virtually "irreversible" insofar as the relaxation time is longer than the seasonal cycle.

b. Effect of upwelling on the QBO

It has not been possible to simulate the quasi-biennial oscillation in Model II with the wave forcings of Model I discussed in section 4 unless \( \bar{F} = 0 \). This difference is not done entirely to coarser vertical resolution of Model II [although some loss of wave amplitude occurs when (3.3) is evaluated numerically on a coarse grid]. Evidently the seasonal cycle, together with the time mean component of tropical heating, induces upwelling at the equator that retards QBO descent. If the equatorial wave forcing is increased, a normal oscillation occurs.

Figure 16 shows the equatorial time–height cross section of zonal wind obtained with wave forcings

\[
\begin{align*}
\kappa_{KQ} &= 25 \text{ m s}^{-1} \\
\kappa_{KQ} &= 2 \\
B_{KQ}(17) &= 10.8 \times 10^{-3} \text{ m}^2 \text{ s}^{-2} \quad (5.2) \\
\kappa_{KT} &= 15 \text{ m s}^{-1} \\
\kappa_{KT} &= 1 \\
B_{KT}(17) &= 2.7 \times 10^{-3} \text{ m}^2 \text{ s}^{-2} \quad (5.3) \\
\kappa_{RG} &= -35 \text{ m s}^{-1} \\
\kappa_{RG} &= 4 \\
B_{RG}(17) &= -6.8 \times 10^{-3} \text{ m}^2 \text{ s}^{-2} \quad (5.4)
\end{align*}
\]
and one-sided friction
\[ \bar{u}_m = 30 \text{ m s}^{-1} \quad (5.5) \]
\[ \alpha_m = \begin{cases} 
(2.5 \text{ d})^{-1} & \text{if } \theta > 0 \\
(10 \text{ d})^{-1} & \text{if } \theta < 0 
\end{cases} \quad (5.6) \]

Together with background friction and Newtonian cooling as in Fig. 3, Hemispheric asymmetry has been introduced in (5.6) to mimic stronger wave drag in the northern winter. As a result, the polar night jet is stronger in the Southern Hemisphere; conversely, tropical easterlies are stronger in the northern winter (the “first” SAO cycle). The period of the QBO is 25 months. Westerlies do not reach above the stratopause, nor does the QBO in general, as expected from scale-dependent Newtonian cooling (Hamilton 1981) and (what is equally important) SAO easterlies. There are no time-mean westerlies in the equatorial mesosphere.

The value of \( \bar{u}_m \) has been chosen to exceed the maximum QBO westerlies so there is no direct effect of friction in the equatorial stratosphere whatever. A companion simulation, shown in Fig. 17, illustrates an indirect effect of friction on the QBO. For this experiment,
\[ B_{KQ}(17) = 9.6 \times 10^{-3} \text{ m}^2 \text{ s}^{-2} \]
\[ B_{KT}(17) = 2.4 \times 10^{-3} \text{ m}^2 \text{ s}^{-2} \]
\[ B_{RG}(17) = -6.0 \times 10^{-3} \text{ m}^2 \text{ s}^{-2} \quad (5.7) \]

and the one-sided friction was reduced to
\[ \bar{u}_m = 30 \text{ m s}^{-1} \quad (5.8) \]
\[ \alpha_m = \begin{cases} 
(5 \text{ d})^{-1} & \text{if } \theta > 0 \\
0 & \text{if } \theta < 0 
\end{cases} \quad (5.9) \]

Other parameters \( c_{KQ}, s_{KQ}, \) etc. were the same as above. Consistent with the reasoning of section 2, the reduced friction is apparent in semiannual easterlies that are weaker than before. But the period of the QBO has now decreased to 23 months in spite of the fact that the equatorial wave forcings were reduced in magnitude. A reduction in forcing per se would normally increase the period of the QBO (Plumb 1977). When reduced friction is applied with the wave parameters of Fig. 16 the QBO period is 19 months (not shown).

What appears to be happening is that the thermal equilibrium of Fig. 2 induces upwelling in the tropics. This nearly inviscid Hadley cell has a small but noticeable effect on the QBO. When friction is introduced, the upwelling increases and the QBO is more severely affected. Figures 18a,b compare vertical velocities at the equator for the simulations of Figs. 16,17; increased upwelling is evident in the simulation with stronger one-sided friction. The semiannual cycle is present, but the time-mean component is more important in the
equatorial lower stratosphere. In the QBO switching region (near 21 km), the vertical velocity approaches 0.2 mm s$^{-1}$, or about 0.5 km month$^{-1}$—a value comparable to what one might infer from Dopplick's (1979) heating rates. This upwelling by itself would not prevent QBO descent, but when the time mean component is added to the QBO easterly shear zone, the net diabatic upwelling can approach or even exceed the rate of descent due to wave forcing. Westerly descent is likewise retarded, although in this case the positive QBO perturbation reduces the departure from radiative equilibrium, so the net effect is to reduce the self-propagation discussed in section 4.

Some effect of horizontal advection is also seen in these simulations, primarily in the west phase. The east phase has a small angular momentum gradient and is fairly insensitive to horizontal advection.

Analysis of the momentum budget in the QBO region supports this interpretation of the oscillation period. Figure 19 displays the body force, vertical advection, and horizontal advection for a simulation with wave parameters (5.7) but stronger friction as in (5.5–6). The oscillation period is about 34 months in this case. There is a large degree of cancellation between body force and vertical advection in the descending easterlies; the net tendency is weaker than what would arise from the body force alone. Horizontal advection is much weaker and closely tied to the seasonal (semiannual) cycle. Note that it is slightly stronger in the west phase. On the other hand, vertical advection is due essentially to the QBO plus time mean and is stronger in the descending easterlies.

When $T^e = 0$, such large values of wave forcing do indeed produce rapid QBO's. An example is shown in Fig. 20 with

$$B_{KQ}(17) = 8.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2}$$
$$B_{KT}(17) = 2.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2}$$
$$B_{RG}(17) = -5.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2}$$

and other parameters as before. The period of the QBO is 14 months in this case.

Table 1 summarizes the results of many QBO simulations with a variety of forcings and one-sided friction. For these experiments the ratio of wave forcings $B_{KQ}(17):B_{KT}(17):B_{RG}(17)$ was held approximately constant, as above. Other parameters $c_{KQ}, s_{KQ},$ etc. were identical. As the one-sided friction is increased, the QBO period increases or simply evolves to a steady state. Larger forcings compensate for increased friction.

---

* There is no seasonal synchronization of the model QBO. The possibility of seasonal modulation, without exact synchronization, is discussed further by Dunkerton (1990).
The threshold for a steady state increases with friction. It is emphasized again that this one-sided friction has no direct effect in the tropics.

The need for increased equatorial wave forcing was implicitly recognized by Gray and Pyle (1989), but has remained puzzling up to this point. (The exact magnitudes in Table 1 may be an artifact of Model II.) The observed equatorial waves no longer seem sufficient for the QBO. They are likely to be essential to the oscillation, but are supplemented by other, unobserved waves. In addition to equatorially trapped waves (Lindzen and Tsay 1975) and planetary Rossby waves (Andrews and McIntyre 1976), the entire spectrum of internal gravity waves $-30 < c < 20 \, \text{m s}^{-1}$ experiences critical level absorption at one time or another in the QBO (Dunkerton 1982a). Their contribution to the QBO should be investigated.

c. Effect of the QBO on the SAO

Model II automatically generates a semiannual oscillation (SAO) due to advection of angular momentum at the equator. One may refer to this kind of oscillation as the “advective SAO.” It was obtained by Holton and Wehrbein (1980) in a similar two-dimensional model. The advective SAO has no westerlies, and may or may not have realistic easterlies; depending on the meridional transport and background friction. The Holton–Wehrbein advective SAO was unrealistically high and weak, due to the high altitude of Rayleigh friction and rather weak meridional flow. Model II generates more realistic easterlies, at least near the solstices. However, the descent of SAO easterlies cannot be simulated without taking into account Rossby wave drag in connection with polar warmings. A more complete study of the SAO using this model must await a parameterization of Rossby wave momentum transport, currently under development by the author. As noted later in text, it will also be necessary to represent the gravity wave spectrum more realistically than simple Rayleigh friction.

Nevertheless, there is one observation that can be made with Model II. Simulations of the stratopause SAO without the QBO are unrealistic when the Hirota (1978) Kelvin wave is included (Takahashi 1984). One may refer to Takahashi’s oscillation as the “inverted SAO” in which easterlies reach their maximum above middle stratosphere westerlies—a pattern opposite to
that observed. The cause of the inverted SAO is an easterly momentum deficit in the middle stratosphere; there are no forces to counter westerly wavemdriving by the Hirota (1978) Kelvin wave. Takahashi (1984), in fact, recognized the need for an easterly momentum source at these levels.

The problem can be partly solved by including the QBO, which on account of its east–west asymmetry induces a time mean easterly flow in the middle stratosphere. Figure 21 shows a Model II simulation in which

\[ c_{KQ} = 25 \text{ m s}^{-1} \]
\[ s_{KQ} = 2 \]
\[ B_{KQ}(17) = 9.6 \times 10^{-3} \text{ m}^2 \text{s}^{-2} \]  
\[ c_{KT} = 15 \text{ m s}^{-1} \]
\[ s_{KT} = 1 \]
\[ B_{KT}(17) = 2.4 \times 10^{-3} \text{ m}^2 \text{s}^{-2} \]  
\[ c_{RG} = -35 \text{ m s}^{-1} \]
\[ s_{RG} = 4 \]
\[ B_{RG}(17) = -6.8 \times 10^{-3} \text{ m}^2 \text{s}^{-2} \]  

and the one-sided friction is

\[ \bar{u}_m = 30 \text{ m s}^{-1} \]  
\[ \alpha_m = \begin{cases} 
(2.5 \text{ d})^{-1} & \text{if } \theta > 0 \\
(10 \text{ d})^{-1} & \text{if } \theta < 0.
\end{cases} \]  

The background friction for this experiment was slightly modified so that the mesospheric flow relaxes to a Gaussian-shaped westerly jet of magnitude \( \bar{u}_m \) and width 15°, centered on the equator. This was done to crudely represent westerly gravity waves and to avoid damping the contribution from an additional Kelvin wave

\[ c_{KS} = 65 \text{ m s}^{-1} \]
\[ s_{KS} = 1 \]
\[ B_{KS}(17) = 4.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2} \]  

also included in the experiment (Hirota 1978). Referring to Fig. 21 it is apparent that the Hirota–Kelvin wave has generated a realistic SAO about half the time, when the QBO is in its east phase near 35 km. In the QBO west phase, however, the inverted SAO pattern prevails; it is similar to Takahashi’s (1984).
It is noted incidentally that the QBO modulates the overlying SAO west phase in the manner discussed by Dunkerton (1979). Also, the strongest SAO east phase occurs at the onset of the new QBO east phase.

The behavior of the SAO west phase is, to a large degree, controlled by the east phase. The pattern of Fig. 21 can be improved by lowering the easterly maxima and incorporating a spectrum of waves that experience less dissipation in the middle stratosphere than the Hirota Kelvin wave. This point was noted by Dunkerton (1979) although at that time the enhanced Newtonian cooling—responsible for the absorption of

### Table 1: QBO period as a function of friction and equatorial wave parameters

<table>
<thead>
<tr>
<th>Run</th>
<th>Friction</th>
<th>Equatorial waves</th>
<th>QBO period (months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>c61</td>
<td>Fig. 20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c36</td>
<td>*</td>
<td>30 m s⁻¹ (5, ∞)</td>
<td>63 km</td>
</tr>
<tr>
<td>c45</td>
<td>Fig. 17</td>
<td>30 (5, ∞) 63</td>
<td>10.8 2.7 * -5/4/-35</td>
</tr>
<tr>
<td>c34</td>
<td>*</td>
<td>30 (2.5, 10) 63</td>
<td>9.6 2.4 * -6</td>
</tr>
<tr>
<td>c35</td>
<td>Fig. 16</td>
<td>30 (2.5, 10) 63</td>
<td>12.3 * -7.5</td>
</tr>
<tr>
<td>c43</td>
<td>Fig. 21</td>
<td>30 (2.5, 10) 63</td>
<td>10.8 2.7 * -6.8</td>
</tr>
<tr>
<td>c40</td>
<td>*</td>
<td>30 (2.5, 10) 63</td>
<td>9.6 2.4 4/1/65</td>
</tr>
<tr>
<td>c49</td>
<td>*</td>
<td>30 (2.5, 10) 56</td>
<td>9.6 2.4 4/2/50</td>
</tr>
<tr>
<td>c63</td>
<td>*</td>
<td>20 (1, 5) 56</td>
<td>12 3 3/4/50</td>
</tr>
<tr>
<td>c38</td>
<td>*</td>
<td>20 (1, 5) 56</td>
<td>10.8 2.7 3/4/50</td>
</tr>
</tbody>
</table>
Fig. 21. Time-height cross section of mean zonal wind at equator as in Fig. 16, including the Hirota (1978) Kelvin wave.

Fig. 22. As in Fig. 21, but with increased friction and a higher wavenumber Kelvin wave.
this wave—was attributed to photochemical acceleration rather than scale dependence. In Fig. 22, the following wave parameters were used:

\[ \begin{align*}
    c_{KQ} &= 25 \text{ m s}^{-1} \\
    s_{KQ} &= 2 \\
    B_{KQ}(17) &= 12.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2} \\
    c_{KT} &= 25 \text{ m s}^{-1} \\
    s_{KT} &= 1 \\
    B_{KT}(17) &= 3.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2} \\
    c_{RG} &= -35 \text{ m s}^{-1} \\
    s_{RG} &= 4 \\
    B_{RG}(17) &= -7.5 \times 10^{-3} \text{ m}^2 \text{s}^{-2} \\
    c_{KS} &= 50 \text{ m s}^{-1} \\
    s_{KS} &= 4 \\
    B_{KS}(17) &= 3.0 \times 10^{-3} \text{ m}^2 \text{s}^{-2}
\end{align*} \]  

(5.17) (5.18) (5.19) (5.20)

and the one-sided friction was increased to

\[ \tilde{u}_m = 20 \text{ m s}^{-1} \]  

(5.21)

\[ \alpha_m = \begin{cases} 
    (1.0 \text{ d})^{-1} & \text{if } \theta > 0 \\
    (5.0 \text{ d})^{-1} & \text{if } \theta < 0.
\end{cases} \]  

(5.22)

Finally, the center of the background friction profile \( \alpha_M \) was lowered from 63 to 56 km. Increased equatorial wave forcings compensate for increased friction (note also that the Parker Kelvin wave has been replaced by an additional Wallace–Kousky wave). The correct time-mean flow in the middle stratosphere is now obtained. It should be noted that the extratropical flow generated by this drag configuration is substantially weaker than before with polar night jet maximizing at approximately 40 m s\(^{-1}\) (not shown). The wind profile, in the stratosphere at least, is similar to the climatological flow of late January or February—after Rossby waves have begun to erode the jet. On the other hand, the climatological mesosphere reaccelerates in February due to mesospheric cooling—an effect that cannot be represented here, but which may be important to the equatorial SAO.

It will be desirable in the future to include a more realistic momentum transport by Rossby and gravity waves. The choice of KS wave parameters in Fig. 22 is based partly on the conclusion by Hamilton and Mahlman (1988) that a broad zonal wavenumber spectrum contributes to the stratopause SAO (their simulated spectrum included waves 1–20, peaking near \( s = 4–7 \), with phase speed \( c \sim 55 \text{ m s}^{-1} \)). Their result is consistent with Hitchman and Leovy’s (1988) observation of a possible deficit in wave driving by ultralong Kelvin waves. It is likely that the Hirota (1978) Kelvin wave experiences strong radiative and perhaps mechanical dissipation in the upper stratosphere. But to model the spectrum adequately, more than a single component should be included or a different scheme should be devised. It would also be appropriate to model the upper mesosphere, including the mesopause semiannual oscillation, without a friction layer.

6. Conclusion

Finite displacement of angular momentum occurs in the tropical middle atmosphere in connection with the quasi-biennial and semiannual oscillations. In the QBO the west phase is relatively narrow due, in part, to localized subsidence at the equator. By contrast, the east phase is very wide, due partly to angular momentum transport in the outflow layer above the easterly shear zone. Nonlinear advection introduces asymmetry in the QBO not present in the linear-diffusive model. It was noted that the exact evolution from initial westerly shear leads to a “solitary” westerly jet and steepened vertical shear. Diabatic advection per se is inadequate for the QBO west phase not because of shear zone decay, but because of angular momentum conservation. Conservation requires the jet to contract, reducing the temperature perturbation and descent rate to unrealistically small values. When Kelvin wave driving is included, the width of the jet is maintained by vertical transport, and descent is accelerated by diabatic advection. The effect has been demonstrated with equal and opposite wave forcings. These results also pertain to simulated quasi-biennial oscillations in a very high resolution two-dimensional model.

The east phase of the stratopause semiannual oscillation appears to be driven, to a large degree, by horizontal advection (cf. Hamilton 1986). A simple nonlinear theory of the middle atmosphere Hadley circulation demonstrates that horizontal advection is strongly influenced by remote body forces in the winter hemisphere. This conclusion is unaltered when a mesospheric friction layer is included, except that the depth of thermal forcing per se is no longer the effective circulation depth. The altitude of a saddle point in the angular momentum distribution is instead determined by body forces, including those in the winter stratosphere due to Rossby wave transport.

The strength of the Hadley circulation is governed primarily by extratropical body forces. A novel result is that the period of the QBO is sensitive to extratropical drag—an effect communicated by the mean meridional circulation. Realistic drag seems to require that equatorial wave forcings exceed their “classic” values in order to generate an oscillation with the observed period (or, in some cases, to generate an oscillation at all). It was suggested that a spectrum of waves, including internal gravity waves, may contribute to the QBO accelerations. It should be emphasized, however, that the net diabatic heating in the equatorial lower stratosphere is a small residual and is not known very
well. Results of Model II should be viewed in this light. Model I is more representative if, perchance, conventional radiative heating rates (Doplick 1979) grossly overestimate the actual upwelling.

In the upper stratosphere, it has been possible to simulate a realistic semiannual oscillation when the QBO is included (with time-mean easterlies in the middle stratosphere) and a spectrum of westerly waves is assumed. Higher wavenumbers experience less absorption, and QBO easterlies in the middle stratosphere (together with horizontal advection) further improve the transmission of these waves. The SAO west phase is, to a large degree, controlled by the east phase. This is one reason why seasonal variation is observed in the SAO (Delisi and Dunkerton 1988). Certain details of the SAO, however, require that Rossby wave transport be represented in a more realistic way. This work will serve as a prelude to future modeling of the SAO with explicit Rossby wave propagation and a spectrum of gravity waves. Hopefully an improved parameterization of equatorial inertial instability can also be devised.

These technical details should not overshadow a more basic theoretical result, viz. the likelihood of “sideways influence” due to body forces adjacent to the tropics. It applies equally well to the troposphere (Held and Phillips 1989). While there is currently much interest in air–sea interactions and their direct role in the tropospheric Hadley circulation, the remote influence of extratropical forces should not be overlooked. It is well known that differences in eddy heat and momentum transport are a major cause of hemispheric asymmetries in the extratropical terrestrial atmosphere. It seems less well appreciated that these forces indirectly affect the Hadley cell and contribute to the hemispheric asymmetry of the tropical circulation.  

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