Radiating and Nonradiating Modes of Secondary Instability in a Gravity-Wave Critical Layer

TIMOTHY J. DUNKERTON AND ROBERT E. ROBINS
Northwest Research Associates, Bellevue, Washington

(Manuscript received 3 July 1991, in final form 25 February 1992)

ABSTRACT

This paper presents results obtained with high-resolution numerical models of the gravity-wave critical layer. The structure and growth rates of preferred modes of secondary instability—within or near regions of potential temperature overturning in the wave field—are discussed. Model instabilities, which appear to be primarily convective, are of two kinds. The expected mode of convective instability is nonradiating, trapped within the region of overturning. A new "radiating" mode of instability was also obtained that has a preferred zonal scale, grows to observable amplitude prior to the nonradiating mode, and extends into the adjacent stable regions of the wave field. As a result, this mode is important in the transition to turbulence and may affect momentum deposition and turbulent mixing due to gravity-wave breaking.

1. Introduction

Internal gravity waves are known to be important in the transport of momentum, heat, and constituents in the terrestrial atmosphere. These waves retard the flow in and above jet streams, especially in the mesosphere, and assist in forcing tropical mean-flow oscillations. They also cause turbulent mixing. Momentum deposition and turbulence is attributable to the unstable breakdown of internal gravity waves propagating vertically in a quasi-compressible atmosphere or approaching a critical layer where wave phase speed equals mean flow speed in the direction of wave propagation. Large-amplitude waves cause isentropic surfaces to overturn—a situation that, in a nearly inviscid flow, is highly unstable. Simple calculations demonstrate that local convective instabilities grow rapidly compared to the time scale for wave propagation, especially in the hydrostatic limit (e.g., Dunkerton and Fritts 1984; Hines 1988). Thus, it is commonly assumed that instabilities will develop, and moreover, be strong enough to "saturate" the large-scale wave field (Lindzen 1981; Dunkerton 1982; Holton 1983; Fritts 1984; Lindzen 1988; Schoeberl 1988; Dunkerton 1989). A central feature of convective-saturation theory is that primary wave amplitude not be allowed to greatly exceed its marginally stable value. The amount of supersaturation varies from model to model, with minor consequences for momentum deposition. On the other hand, turbulent diffusivity of the mean state is sensitive to the wave amplitude and degree of turbulence localization within the wave field (Fritts and Dunkerton 1985; Coy and Fritts 1988; Dunkerton 1989; McIntyre 1989).

Because little is actually known about the unstable breakdown of gravity waves, compared to what is assumed by theory, the gravity-wave secondary instability is an appropriate topic for direct numerical simulation and laboratory study. The problem is only beginning to be investigated. Numerical simulations of gravity-wave breakdown were recently performed in a compressible atmosphere by Walterscheid and Schubert (1990) and in a Boussinesq fluid by Winters and d'A-saro (1989). Laboratory experiments simulating the gravity wave—critical layer interaction were undertaken by Koop and McGee (1986) and Delisi and Dunkerton (1989). All of these studies demonstrated that a large-amplitude gravity wave, with steepened or overturned isentropic (isopycnal) surfaces, is unstable to smaller-scale motions; these instabilities quickly become turbulent, and the entire process causes momentum deposition in the region of breakdown. To this extent, the laboratory and numerical evidence lends support to saturation theory. There are, however, unknowns. Evidence of constituent mixing, for example, is unclear; laboratory experiments of Delisi and Dunkerton (1989) indicated only a modest rearrangement of the initial stratification by persistent wave breaking lasting several hours (see also Delisi and Orlanski 1975). It should be kept in mind that these experiments had essentially monochromatic wave forcing; turbulence was observed only within or near regions of steepening...
and overturning. Given the sensitivity predicted by theory, experimental results may not apply to all cases. More disturbing is that there is as yet no agreement on the dominant mechanism of secondary instability within the primary wave field, nor has it been possible to explicitly represent the saturation process itself. In Winters and d’Asaro, periodic boundary conditions were assumed with an initial condition, but no boundary forcing. As the wave packet approached the critical level, it became unstable, apparently via shear instability, to small-scale turbulent motions. The experiment was designed so that a turbulent cascade would eventually be absorbed by scale-dependent damping. Thus, the primary wave was simply absorbed; no equilibration process could be represented in their formulation. In Walterscheid and Schubert, the initial evolution of convective instabilities was simulated, but computational constraints prevented a long-time integration much beyond the onset of instability. Their results did suggest at least an initial equilibration of primary wave energy due to secondary instability. Laboratory experiments of Delisi and Dunkerton (1989), which could be run for very long times, indicated that quasi-equilibrated, “saturated” states may be possible. Their structure, however, resembled a turbulent hydraulic jump, which might be an artifact of tank geometry and experimental conditions. In any case, the wave forcing was eventually altered by the interaction, making comparison to saturation theory difficult.

There is a valuable contribution from these studies, the observation that more than one type of secondary instability is possible. The initial instability in Delisi and Dunkerton resembled an inclined row of Kelvin–Helmholtz billows. On the other hand, small-scale features in Walterscheid and Schubert resembled cellular convection. Theory indicates a preference for Kelvin–Helmholtz or convective instability depending on wave and mean-flow parameters (Dunkerton 1984, 1989; Fritts and Rastogi 1985; Fritts and Yuan 1989; Yuan and Fritts 1989). Breaking in Walterscheid and Schubert was due mainly to ambient density decrease, while mean flow shears were initially present and rapidly intensified in laboratory experiments of Delisi and Dunkerton. The variety of instabilities makes it necessary to undertake a detailed investigation of gravity-wave breakdown in order to understand its effect on momentum deposition, constituent mixing, and saturation.

As part of an ongoing study, we are using direct numerical simulation with a hierarchy of models to investigate the propagation and unstable breakdown of internal gravity waves in the atmosphere. The waves have horizontal scales ranging from about 50 km, thought to dominate the mesospheric momentum budget, extending to synoptic and planetary scales as inertia–gravity waves and equatorially trapped planetary waves important in the upper troposphere and stratosphere. The investigation reported in this paper utilized hydrostatic and nonhydrostatic models to simulate two-dimensional gravity-wave propagation and breakdown in a stratified fluid without rotation. For simplicity, a monochromatic forcing was assumed, with primary wave propagating vertically into spatially and temporally constant linear mean-flow shear having a critical level in the upper part of the domain. This configuration is similar to earlier studies (e.g., Fritts 1982, 1985; Dunkerton and Fritts 1984; Fritts and Dunkerton 1984; Dunkerton 1987), but horizontal resolution is now much finer than before. Here, an attempt was made to simulate a two-dimensional fluid, including turbulent breakdown, and not merely the interaction of a few large-scale waves. As will be shown, transition to a turbulent state includes quasi-circular motions in the plane of propagation with a length scale defined, in part, by the depth of overturning. Below the critical level, this scale can be shallow, so high horizontal resolution is required.

This requirement was met by Walterscheid and Schubert (1990), allowing secondary instabilities to be simulated. In addition, their paper reviewed many fundamentals of gravity wave–mean flow interaction. Here, we endeavor to build on their results with the help of simpler models, in order to examine in detail the spectral evolution and structure of local secondary instabilities in a gravity-wave critical layer. The problem turned out to be more interesting than expected, as there are several modes of instability in an overturned gravity wave. Because the basic state consisting of primary wave and zonal mean flow is convectively unstable, and is more complex than assumed by Yuan and Fritts (1989), the instability problem is more complicated. A new result from our study is a “radiating” mode of secondary instability that develops within the region of overturning, deriving energy from the unstable stratification, while simultaneously extending into the adjacent stable regions of the wave field. Because of its preferred stable horizontal scale, this mode can play a role in the transition to turbulence.

In section 2, the numerical models are described. Section 3 outlines the basic simulation and presents results obtained with a nonlinear hydrostatic model. In section 4, comparison is made to linear nonhydrostatic models, including theoretical discussion of convectively unstable modes obtained in a parallel-flow approximation. Some implications of these results are mentioned in section 5.

---

1 Yuan and Fritts (1989) and Fritts and Yuan (1989) determined growth rates of Kelvin–Helmholtz instability in an inertia–gravity wave with and without mean shear, respectively, for convectively stable flow approximated by hyperbolic, rather than sinusoidal, functions.
2. Numerical models

a. Nonlinear hydrostatic model

The nonlinear model began with hydrostatic equations of motion in log-pressure coordinates, that is,

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial x} = X \]  \hspace{1cm} (2.1)

\[ \frac{\partial \phi_z}{\partial t} + u \frac{\partial \phi_z}{\partial x} + w \left( N^2 + \frac{\partial \phi_z}{\partial z} \right) = Q \]  \hspace{1cm} (2.2)

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0, \]  \hspace{1cm} (2.3)

where \( u \) and \( \phi \) are zonal velocity and geopotential, respectively, \( w \) is vertical velocity (in log-pressure coordinates; see Holton 1975), \( z \) is log-pressure height, and \( H \) is the density scale height. Equations were divided into zonal mean and perturbation, to be solved separately. For simulations reported here, the mean flow was held constant in time; the effect of mean-flow acceleration is examined in the Appendix (see also Dunkerton and Robins 1992). The entire flow was assumed two-dimensional, so that meridional velocity \( v \) and any \( y \) variations of the flow were ignored, as was rotation \( (f = 0) \). This left \( u \) (zonal wind) and \( \phi_z \) ("temperature") as dependent variables of the model. Equations for \( u \) and \( \phi_z \) were stepped forward in time using a second-order Adams-Bashforth method.

The next step was to solve for \( \phi \) and \( w \), accomplished by vertical integration using Simpson's rule. The integration, and evaluation of horizontal derivatives, was done in zonal wavenumber space using a fast Fourier transform between physical and spectral space. To avoid aliasing, the top one-third of the zonal wavenumber spectrum was set equal to zero at each time step. A nearly monochromatic primary wave was excited at the lower boundary by specifying \( w \) there. A profile of vertical velocity was obtained by integrating the continuity equation to the upper boundary, where the Klemp and Durran (1983) radiation condition was imposed, giving \( \phi \) at that point. The geopotential profile was then obtained by downward integration of \( \phi_z \). To avoid numerical noise at the lower boundary, nonlinear terms were artificially weighted by a sine ramp function in the lowest one-tenth of the model domain.

Various model resolutions were used. For simulations reported here, 201-401 vertical grid points gave nearly identical results, allowing the lower resolution (201) to be used for most of the nonlinear runs. Horizontal resolution included 256 zonal harmonics. This relatively high resolution, however, was necessary only to represent secondary instabilities. It was acceptable to force the primary wave at lower resolution (128 harmonics) until instabilities first became visible in the spectrum (typically, about 20 orders of magnitude below the primary wave). At that point, we zero filled the zonal wavenumber spectrum between harmonics 129 and 256, giving a complete spectrum of waves 1–256 for use as an initial condition in 256 harmonic runs. In retrospect, this procedure would have been an economic necessity. Fortunately, it was completely adequate for the simulations reported here.

Terms on the rhs of (2.1)–(2.2) represent dissipation in the form of scale-dependent diffusive damping (described later); they were set to zero in inviscid simulations.

b. Linear model of nonhydrostatic nonparallel flow

Dunkerton (1987) used a low-resolution two-dimensional model of nonhydrostatic flow to simulate the nonlinear interaction between large-scale gravity waves via parametric subharmonic instability. As described in that paper, the nonhydrostatic model used a streamfunction–vorticity representation with vorticity and potential temperature (or buoyancy) as dependent variables.

The same model could be used here to study the linear evolution of secondary instabilities in a gravity-wave critical layer so that comparison could be made to the nonlinear hydrostatic model of section 2a. As originally formulated, Dunkerton's model (1987) calculated nonlinear terms in spectral space without transforms; it was then possible to devise a linear model by retaining only certain nonlinear interactions. Denoting the primary wave as "wave 1," the linear model assumes that (i) each nonlinear product of spectral coefficients includes wave 1 as one of the components of the product (rather than a sum over all wavenumbers as in the original nonlinear model) and (ii) all nonlinear contributions to wave 1 can be neglected. These criteria approximate a fully nonlinear model when all harmonics \( N > 1 \) have small amplitude. The advantage of a linear model is that the modal structure of secondary instabilities can be isolated without disturbing the basic state (zonal mean plus primary wave) once the instabilities attain finite amplitude.

In practice, secondary instabilities evolve on a much faster time scale than the primary wave, so it was acceptable (for qualitative purposes) to freeze wave 1 at some initial condition. This was determined by integrating a single-harmonic model until overturning. It was unnecessary to include wave 1 in the secondary instability, due to separation of scales.

c. Linear model of nonhydrostatic parallel flow

Scale separation suggests that a parallel-flow approximation could be useful for interpretation. This approach was used by Fritts and Yuan (1989) and Yuan and Fritts (1989) to determine growth rates of Kelvin–Helmholtz instability in an inertia–gravity wave.
A parallel-flow model used columns of "basic state" zonal wind and temperature obtained from the two-dimensional model of section 2b. That is, the flow in any column was assumed independent of horizontal direction $x$. Unstable modes were isolated by solving the one-dimensional Taylor–Goldstein equation

$$\psi_{zz} + \psi \left[ \frac{N^2}{(c - \bar{u})^2} + \frac{\bar{u}_{zz}}{c - \bar{u}} \right] - k^2 = 0, \quad (2.4)$$

where $\psi$ is disturbance streamfunction, $c$ is complex phase speed, $k$ is zonal wavenumber, and $N^2$, $\bar{u}$ represent static stability and zonal wind of the basic state (now distorted by a finite-amplitude primary wave) that, in accord with the parallel-flow approximation, was assumed independent of $x$. To locate complex eigenfrequencies a shooting method was used like that of Dunkerton (1990) with boundary conditions $\psi = 0$.

3. Results of nonlinear hydrostatic model

a. Primary wave

For the case reported here, domain depth was 10 km, with linear $\bar{u}$ profile varying from $-16$ m s$^{-1}$ at the lower boundary to $+4$ m s$^{-1}$ at the upper boundary. Background stratification $N^2 = 0.0004$ s$^{-2}$, implying for isothermal conditions a density scale height $H = 7014$ m. Domain width was 50 km, also the horizontal wavelength of primary wave (wave 1). A stationary, nearly monochromatic disturbance was excited by specifying vertical velocity of wave 1 at the lower boundary with amplitude 0.4 m s$^{-1}$, increasing gradually from zero to its full value at $t = 10000$ sec (held constant thereafter). The critical level for this wave was at 8 km. The model time step was $\Delta t = 0.0625$ sec. Our first calculation was inviscid, that is, $X = Q = 0$.

The resulting evolution of potential temperature is shown in Fig. 1 at various times. Overturning began just prior to 15 000 sec; however, secondary instabilities did not become visible in physical space until somewhat later, after 19 000 sec. Until this time, structure of the primary wave determined by the hydrostatic model was virtually identical to the linear nonhydrostatic model, consistent with the shallow inclination of this wave ($\sim 30:1$ in the region of overturning) and small amplitude of higher harmonics.

Perturbation temperature is shown in Fig. 2 for times corresponding to Fig. 1. Vertical wavelength of the primary wave contracted approaching the critical layer; in the region of strongest overturning or unstable lapse rate, it was $\sim 1.5$–2 km.

b. Spectral evolution

Figure 3 shows the development of the vertical component of perturbation kinetic energy, $\langle w'^2 \rangle$, integrated over the entire domain, as a function of zonal wavenumber, for the inviscid simulation of Figs. 1 and 2. The spectrum evolved in two phases. In the first phase, energy appeared in a distinct peak near zonal wavenumber 48; this peak moved to lower wavenumbers with time. Meanwhile, energy appeared at higher wavenumbers with no distinct zonal scale. This region of the spectrum will be referred to as the "convection continuum." In the second phase, after 17 000 sec, the peak near wavenumber 32 was overtaken by the continuum and its identity lost. By 18 000 sec, the vertical component of kinetic energy beyond about wavenum-
were run through a high-pass filter retaining all zonal wavenumbers greater than 20. Figure 4 shows the resulting temperature field at 18,000 sec, with two velocity vectors superposed, in a region of the primary wave where instabilities attained large amplitude (near the location of strongest overturning). As implied by Fig. 3, there was a mix of zonal scales, but dominant scales were near the resolution of the grid. This can be seen in the temperature perturbations, which had vertical wavelengths as small as 100 m (2 grid points). Counting wavelengths from left to right, the dominant zonal wavenumber was about 150, near the truncation point (see Fig. 3). In all inviscid cases performed, there was no horizontal scale selection in the convection continuum apart from numerical truncation of the model. In effect, the zonal scale of instability was determined at the outset by the number of harmonics retained in the calculation.

In contrast, the vertical scale of streamfunction or velocity perturbations was defined by the depth of overturning. In physical space, convection cells were elongated in the vertical. This is expected to be the most efficient rearrangement of parcels necessary to stabilize the large-scale wave field (Dunkerton 1989). It was only within the region of overturning that vertical velocities of the secondary instability had significant amplitude.

Similar reasoning suggests that the convection continuum will, in fact, be three dimensional. Thus, it cannot be represented adequately in a two-dimensional model. For our purpose, this limitation is not critical, as the energy cascade is assumed to proceed from low to high wavenumbers and there undergo irreversible absorption. More relevant is whether the initial insta-

---

**Fig. 2.** Perturbation temperature for simulation of Fig. 1 at the same times. Contour interval: 0.067 K. Warm (cold) regions shown in white (black).

---

**Fig. 3.** Spectrum of the vertical component of perturbation kinetic energy as a function of zonal wavenumber for inviscid simulation. Labels denote time in seconds × 1000.
bility is properly represented. The "radiating" mode (discussed next) depends on variations of primary wave zonal velocity in the vertical plane of propagation; hence, it is arguable that the 2D model captured, or at least approximated, the initial mechanism of gravity wave breakdown. Laboratory instabilities appear initially two dimensional as well (Delisi and Dunkerton 1989), although they are probably more closely related to shear instability of the primary wave and the mean flow.

d. Scale-dependent damping

Vertical velocity spectra in the inviscid simulation displayed an initial peak near wavenumber 48 that was later overtaken by convection at higher wavenumbers. As a result, the identity of this peak was eventually lost. Curious to know more about it, we performed a second simulation (beginning at the zero-fill point, $t = 14 \, 000$ sec) with scale-dependent damping included. The damping was Laplacian with the diffusion coefficient depending on zonal wavenumber (a hyperbolic tangent centered on wavenumber $k = 96$ having width $\Delta k = 4$).

The magnitude of the coefficient asymptoted to a maximum $20 \, \text{m}^2 \, \text{s}^{-1}$ at the highest wavenumbers. This maximum value was increased to $40 \, \text{m}^2 \, \text{s}^{-1}$ after $18 \, 000$ sec to compensate for growth of convection; it was doubled again at $19 \, 000$ and $20 \, 000$ sec.

Temporal development of the vertical velocity spectrum with scale-dependent damping is shown in Fig. 5. Unlike the inviscid case, the convection continuum was suppressed until well after $19 \, 000$ sec; instead, there were several peaks in the spectrum at approximately equal intervals. Formation of secondary peaks is not well understood, although it seems to be due to an initial doubling of the first peak near wavenumber 32 (whereupon the first and second peaks interact to produce the third, and so on). At a later time individual peaks were filled in by the continuum.

e. Structure of radiating mode

Temperature fields associated with the first and second spectral peaks are shown in Figs. 6a,b in a region where these fields had relatively large amplitude. Structure of the first peak consisted of a row of cells within the region of overturning (between 4.25 and 5

![log E_w](image)

**Fig. 5.** Spectrum of the vertical component of perturbation kinetic energy as a function of zonal wavenumber for simulation with scale-dependent damping. Labels denote time in seconds $\times 1000$. 
km) with clockwise motion about warm centers and vice versa. In the adjacent stable regions of the wave field, separate cells appeared to “radiate” away in both directions, with larger amplitude in the lower lobe. As it turned out, this three-lobed structure was a characteristic feature of a radiating mode of instability with preferred zonal scale in nonhydrostatic models discussed in the next section. Unlike the convection continuum observed in the inviscid simulation, velocities in Fig. 6a had significant amplitude outside the region of overturning.

The second spectral peak, with structure shown in Fig. 6b, shared this radiating property to some extent but, due to its smaller zonal scale, also resembled the elongated convective cells seen in Fig. 4. Compared to Fig. 6a, there was smaller phase tilt in the lower lobe, suggesting a more evanescent structure there.

4. Results of nonhydrostatic models

Although the primary wave was essentially hydrostatic, there was no guarantee that secondary instabilities would also be hydrostatic. It was important, then, to perform nonhydrostatic model simulations. Apparent from these models was that both radiating and nonradiating modes exist with structures similar to those obtained in the nonlinear hydrostatic model. [See also Dunkerton and Robins (1992) and the Appendix for further discussion of nonlinear nonhydrostatic simulations.]
vertical scales of instability were somewhat larger. This made simulation more economical by reducing the time required and allowing a truncated zonal wavenumber spectrum (waves 1–64) to be used. As it turned out, the radiating mode of instability peaked near wavenumber 22, well within resolution. Growth of nonradiating convection, on the other hand, increased with wavenumber all the way to the end of the resolved spectrum, as in the nonlinear hydrostatic model. Peak rate was observed near wavenumber 64, that is, the truncation point.

Growth rates of radiating and nonradiating modes were 0.006476 and 0.01444 s$^{-1}$, respectively, although for the first several hundred seconds of simulation, the radiating mode was everywhere larger than the nonradiating mode. This result is attributed to a cascade of energy from the primary wave that would cause intermediate zonal wavenumbers to grow first, despite having a smaller growth rate. Even at later times, near 25 000 sec, the radiating mode was visible outside the region of overturning. By this time the nonradiating mode became dominant within the region of overturning, at the smallest resolved scales of the model (near wavenumber 64).

To isolate radiating modes, the simulation without scale-dependent damping was repeated with such damping designed to suppress all zonal wavenumbers greater than 36, while leaving smaller wavenumbers unaffected. In this case, the nonradiating mode was eliminated, and the same radiating mode was obtained. Figures 8a,b show the temperature and streamfunction fields at $t = 25 000$ sec (after 1000 sec of simulation) in the region of strongest overturning denoted by a box in Fig. 7. The shallow inclination of the primary wave can be seen, slanting upwards from left to right. Maximum amplitude of the radiating mode was in the center of overturned region, with a slightly smaller maximum in the stable region immediately below. Its amplitude was considerably smaller in the stable region immediately above. In both stable regions, phase lines tilted forward, going away from center.

Similar remarks apply to streamfunction, except for left–right asymmetry within individual cells, particularly in the center lobe. There was greater upwelling (downwelling) in the left half of warm (cold) centers. Zonal average through the center lobe gives a net upward heat flux, that is, downgradient with respect to the unstable lapse rate in this region. (This can be seen by overlaying Figs. 8a,b.)

Except for zonal scale (near wavenumber 22), the radiating mode of instability observed here was similar to that of the nonlinear hydrostatic model.

b. Unstable modes of parallel flow

Radiating and nonradiating modes of instability were found in the parallel-flow model (section 2c). In this
case, a parallel flow was constructed using output from the 2D linear model in a column at the center of the domain. Profiles of zonal velocity and potential temperature in this column are shown in Fig. 9. There was excellent correspondence between overturning and positive zonal velocity (as expected from Orlanski and Bryan 1969), indicating that by this time the primary wave had settled down to a nearly stationary phase speed. This was not yet the case at higher levels. Consequently, the small region $\bar{u} > 0$ near $z = 7$ km did not have $\bar{\theta}_z < 0$; there was only one region of overturning in the column. (In this discussion, an overbar now denotes the combined primary wave and zonal-mean state.)

Search for unstable modes of the Taylor–Goldstein equation (2.4) in the frequency range $-0.02 < \omega_r < 0.02$ s$^{-1}$ and for growth rates $0 < \omega_i < 0.02$ s$^{-1}$ revealed several modes of instability. Dominant modes correspond to the nonradiating and radiating modes discussed earlier. We will restrict attention to these two modes. Growth rates ($\omega_i$) and real frequency ($\omega_r$) are shown in Fig. 10 as a function of zonal wavenumber (with respect to the basic horizontal scale, 50 km). There was peak growth near wavenumber 22, corresponding to the radiating mode, and uniform increase of growth rate with wavenumber at higher $k$, corresponding to the nonradiating mode. (Note that the growth rate at wavenumber 64 was smaller than observed in the nonparallel model; this difference is attributed to the artificial zonal truncation of that model.)

Calculated growth rates were qualitatively consistent with the spectral evolution observed in the nonlinear hydrostatic model, increasing with zonal wavenumber. Presumably, in a model with sufficient resolution growth rate would asymptote to its maximum possible value consistent with the unstable stratification [Eq. (4.6)]. Zonal scale selection is not obtained apart from scale-dependent damping.

Near wavenumber 40, growth of the nonradiating mode leveled off; inspection of the solution revealed that the mode trajectory below this point belonged not to the single-lobed, nonradiating mode but rather a “ducted mode” with multiple lobes and maximum
amplitude in one or another of the adjacent stable regions of the basic state. These modes are being investigated further. They appear to have weaker growth rates than either of the two modes discussed here.

Structure of the most unstable radiating mode is shown in Fig. 11, with velocity vectors superposed on potential temperature anomalies. Only a vertical slice of the total field is shown, with aspect ratio unity. As before, the center lobe was largest. The lower lobe was about 85% as large, considerably stronger than the upper lobe. In the center lobe the phase relationship between vertical velocity and temperature implied an upward heat flux. Momentum flux was positive (negative) in the upper (lower) lobes.

Zonal propagation relative to the fluid was consistent with thermodynamic balance. Noting that phase speed relative to the ground was small (about $-1 \, \text{m s}^{-1}$), intrinsic phase speed $\tilde{c}$ was essentially $-\omega$. In the stable sidelobes $\tilde{u} < 0$ so that $\tilde{c} > 0$. In these regions upwelling was to the right of cold centers, causing the disturbance to propagate to the right relative to the fluid. In the region of overturning $\tilde{u} > 0$, so that $\tilde{c} < 0$. Here, the stratification was unstable so that upwelling caused warming; the disturbance then propagated to the left relative to the fluid. It also grew with time, as upwelling slightly overlapped the warm centers and vice versa.

Momentum flux of the radiating mode was downgradient with respect to the basic-state velocity profile, suggesting that shear also played a role in generation of perturbation kinetic energy (PKE). The relative importance of shear and unstable lapse rate is currently under study. Preliminary analysis suggests that the radiating mode could be regarded as a convective instability modified by shear and aided by shear generation of PKE. Amplitude was largest within the region of overturning, where its structure was qualitatively similar to that of nonradiating convective instability.

Structure of the nonradiating mode at wavenumber 64 (not shown) revealed a single-lobed streamfunction confined to the region of overturning. This mode is the conventional convective instability, with increasing growth rate as a function of zonal wavenumber.

c. Scale selection and the hydrostatic approximation

An unresolved issue is the mechanism of scale selection in the nonlinear hydrostatic model. The hydrostatic approximation eliminates any dependence on zonal wavenumber from the Taylor–Goldstein equation (2.4). Selection of a preferred zonal scale (or integral multiples of this scale) was apparently due to zonal variation of the nonparallel quasi-stationary basic state.
5. Discussion

The two-dimensional analysis indicates that convective instabilities are an inevitable consequence of overturning of nearly hydrostatic gravity waves in an inviscid, nonrotating fluid with weak ambient shear. Growth rates of conventional, nonradiating instability appear to be consistent with the unstable stratification, but without zonal-scale selection. This is in contrast to the most unstable radiating mode, with its slightly smaller growth rate and finite zonal scale. Interestingly, the radiating mode dominates the initial evolution to turbulence, due to a cascade of energy from the primary wave. Under the right circumstances, this mode might grow to become the dominant saturation mechanism within the region of overturning. It is, in any case, an important mode of instability extending into the adjacent stable parts of the wave field. Some implications of these results are now briefly discussed.

a. Saturation

It appears that the delay of wave saturation, relative to the first overturning of the primary wave, could be estimated by using as an upper limit the maximum possible growth of convective instabilities. Their growth rate is a function of the unstable stratification within the wave field. It has been generally assumed that this delay is negligible. That this is not the case is clear from simulations discussed earlier (see also Winters and d'Asaro 1989; Walterscheid and Schubert 1990). To the extent that one needs to account for the transient development of saturation, it would be simple to relax the primary wave amplitude to its saturated value with a time constant given by the maximum growth rate of convection—determined by whatever supersaturation is actually achieved in the primary wave. A supersaturated equilibrium state could then be defined as a balance between the supply of wave activity from below (dependent on the group velocity, basic state, and wave forcing) and depletion of primary wave energy due to secondary instabilities (dependent on the degree of supersaturation).

b. Momentum transport

It is well known that if convective instabilities grow at the expense of the primary wave, momentum flux due to this wave will become convergent in the region of breaking, causing mean-flow acceleration (Lindzen 1981; Dunkerton 1982). The effect is due simply to a reduction of primary wave amplitude, and does not take into account any additional contributions from the instabilities themselves. Flux convergence is confined to the region of overturning and wave front. Existence of a radiating mode of instability implies reverse momentum transport in the stable part of the primary wave field immediately below the region of overturning. This would normally have minor consequences, reducing the net momentum flux at a slightly lower level. The effect would be more important if the stable part of the wave field below the region of overturning occupied a significant fraction of the domain. In this case, reverse momentum flux might become visible throughout the lower part of the fluid well before the critical layer descends into this region (perhaps it would also prevent such descent). This possibility merits further investigation in parameter ranges different from those used above.

c. Constituent mixing

If the radiating mode were to grow by itself to finite amplitude, ensuing turbulence would occur in both the stable and unstable regions of the primary wave field. Turbulence would no longer be localized within the region of overturning, implying significant mixing of heat and constituents (Fritts and Dunkerton 1985; Coy and Fritts 1987; Dunkerton 1989).

In reality, radiating and nonradiating modes coexist, and it seems likely that turbulence will be substantially stronger within the region of overturning. There may, however, exist enough turbulence immediately outside this region to enhance the effective diffusivity of the mean state. Beyond this, it is difficult to be specific without considering particular cases. In note incidentally that other instability mechanisms, such as Kelvin–Helmholtz instability, imply negligible diffusivity (Fritts and Yuan 1989; Yuan and Fritts 1989).

6. Conclusions

Two-dimensional hydrostatic and nonhydrostatic models were used to study secondary instabilities in a gravity-wave critical layer. For a nonrotating fluid with weak ambient shear, convective instabilities are preferred and develop at a rate consistent with the unstable stratification. Fastest growth occurs for a nonradiating mode of convective instability confined mostly within the unstable region. In an inviscid fluid there is no preferred finite horizontal scale of instability; model calculations indicated most rapid growth at the smallest resolved scales. A preferred scale is obtained with scale-dependent damping, which could also be used to elim-
minate this mode; such damping was artificial in the parameter range of interest.

Radiating modes of instability were also found that have not, to our knowledge, been documented previously. The most unstable radiating mode has a three-lobed structure, with maximum amplitude in the unstable region and slightly smaller amplitude in the stable region immediately below. Although growth of this mode was somewhat smaller than the nonradiating mode, it was considered important for two reasons. First, the zonal scale is finite and relatively close to the primary wave, so this mode could, for a time, exceed the amplitude of the nonradiating mode even within the region of overturning. Second, of the two modes, the radiating mode is the only one having significant amplitude outside the unstable region. Therefore, it may be important in momentum transport and constituent mixing. The significance of this result is that, until now, it was thought that if convective instability is preferred, turbulence would be confined largely to the overturned region of the wave field, and if this instability prevents any significant supersaturation, the effective diffusivity of the mean state would be weak (Fritts and Dunkerton 1985; Coy and Fritts 1987). Direct simulation indicates, however, that supersaturation is likely initially, and radiating instabilities appear outside the region of overturning.

Theoretical analysis reveals other radiating instabilities that appear as “ducted” modes above and below the unstable region. They have not been discussed here as growth rates are weaker, and, in any case, further investigation of their mode trajectories is necessary.

It is interesting to speculate what would happen if the third dimension could be included in the simulations. It seems likely that the radiating mode of instability will occur in the two-dimensional plane already represented by the model, as the three-lobed structure of this mode is consistent with thermodynamic balance by virtue of alternating signs of ambient zonal velocity in this plane (cf. section 4b). If so, the two-dimensional model has described, at least, the initial evolution to three-dimensional turbulence. For nonradiating convection, on the other hand, there is no reason why instability could not occur in any plane; the transverse plane might be preferred (e.g., Klaassen and Peltier 1985).

With today’s computers, the third dimension is not accessible to direct numerical simulation at the resolution necessary for our case study. It would be more fruitful to pursue this problem using a turbulence closure scheme for nonradiating convection.

Acknowledgments. This research was supported by the Air Force Office of Scientific Research, Contract F49620-89-C-0051, and by the National Science Foundation, Grant ATM-8819582. Partial computing support was also provided by the National Center for Atmospheric Research, which is sponsored by the National Science Foundation.

APPENDIX

Effects of Mean-Flow Modification

The aforementioned results were obtained for fixed mean flow. Similar instabilities were found when mean-flow modification was included, as now shown.

![Evolution of mean flow in experiment with mean-flow modification](image1)

**Fig. 12.** Evolution of mean flow in experiment with mean-flow modification. Units: meters per second. Time in seconds × 1000.

![Perturbation temperature at 21 000 s](image2)

**Fig. 13.** Perturbation temperature at 21 000 s in experiment with mean-flow modification. Units on axes: kilometers. Contours: 0.05 K.
The numerical model used here was a nonlinear, time-dependent version of the linear nonhydrostatic code used in section 4 to study temporal development of secondary instabilities on a fixed basic state. As such, it was identical to Dunkerton's (1987) model except that horizontal resolution was increased and evaluation of nonlinear Jacobians done with an FFT to physical space.

An experiment similar to that of section 3 was done using the nonlinear nonhydrostatic model, with mean-flow modification due to gravity-wave momentum flux convergence and scale-dependent damping (Dunkerton and Robins 1992). Evolution of mean flow is shown in Fig. 12, and was similar to that of Dunkerton and Fritts (1984); that is, formation of a "ledge" at the base of the critical layer and some irregular oscillations above.

Mean-flow modification altered the primary wave structure, reducing vertical wavelength over a substantial part of the domain (see Fig. 13, and compare to Fig. 2). This occurred as a result of reduced intrinsic phase speed due to mean-flow acceleration.

Evolution of the vertical velocity spectrum is shown in Fig. 14. There was a peak associated with secondary instability centered between zonal wavenumbers 9 and 20; it was eventually overtaken by the convection continuum. As in the previous case, the spectrum equilibrated to its final shape before 30 000 sec.

Mean-flow modification altered the dominant
wavenumber of secondary instability but did not prevent its development. Structure associated with secondary instability peak at 21 000 sec is shown in Fig. 15 and in expanded form in Fig. 16. It was quite similar to Fig. 6a, although occurring at a lower altitude where the primary wave was most unstable. This is, of course, consistent with mean-flow modification that lowered the critical layer, and hence lowered the altitude of wavebreaking and turbulence. Similar behavior was observed in laboratory experiments, although the mechanism of secondary instability may have been different in that case (Delisi and Dunkerton 1989).

REFERENCES


