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Existence of strong solutions to the generalized inverse of the quasi-geostrophic equations

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Abstract. Existence of strong (i.e. classical) solutions to the generalized inverse of the three-dimensional quasi-geostrophic equations, describing the large-scale motion of the atmosphere and oceans, is proved for a finite time interval. Both the dissipative and the nondissipative cases are considered. The spatial domain considered is doubly periodic in the horizontal directions and is bounded above and below by rigid, horizontal surfaces. The generalized inverse is defined as the solution to the Euler–Lagrange equations, obtained by minimizing a weighted sum of errors in the quasi-geostrophic equations, boundary conditions, and data, and thus represents a solution that approximates both the equations and the data available inside the domain. The proof relies on the Schauder fixed-point theorem applied to the appropriate Hölder function spaces. The finite time interval over which the proof is valid is not arbitrary, but depends on the norms of the initial conditions in such a way that, as the norms of the initial conditions increase, the time interval decreases.

1. Introduction

The quasi-geostrophic equations, both dissipative and nondissipative, are a widely used approximation to the motion of the atmosphere and oceans, conventionally derived from the primitive equations by asymptotic expansion in small Rossby number [17]. Their domain of applicability is primarily that of synoptic to planetary scale motions in midlatitudes, and their validity, including finite time existence and uniqueness results, has been discussed in detail in, for example, [4–7, 9]. Computationally they represent a significant advantage over the full primitive equations from which they are derived, by only representing the broad scale, Rossby wave mode and filtering the two fine scale, gravity wave modes. They thus represent the essential broad scale, slow time motion of the atmosphere and are an example of balanced dynamics. Conceptually they provide a useful simplification of the full dynamics, and a starting point for further idealizations and analytic investigations.

When such equations are used for data assimilation, undetermined errors are included in the equations and additional data is specified, typically leading to an inverse problem. In the atmosphere and oceans the data can take many forms, from satellite measurements of sea-surface elevation to radiosonde measurements of upper troposphere temperature, and a measurement functional must be specified to relate the particular measured quantity to the appropriate model variables [3]. In general, a solution is sought that approximates both the model equations and the data, or, equivalently, the equations are used to interpolate between the (typically sparse) data values (see, e.g., [3,8,21] for recent overviews of data assimilation in atmospheric and oceanic contexts; also see [19] and other articles in the Meteorological Society

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of Japan special issue on Data Assimilation in Meteorology and Oceanography: Theory and Practice). In variational assimilation the undetermined model and data errors are minimized in a weighted least-squares sense to provide a generalized inverse solution. If the weights in the minimization are chosen as the covariances of the model and data errors then it can be shown (e.g. [3]) that the generalized inverse solution is optimal; under suitable further assumptions about the means and covariances of the model and data error the procedure is equivalent to optimal interpolation. As an example, [8] used a single-layer quasi-geostrophic model together with measurements of streamfunction at model gridpoints to construct inverse solutions in a simple idealized setting.

The generalized inverse solution, if it exists, is generally obtained by solving the Euler–Lagrange (EL) equations that result from the minimization of the model errors and the data errors (here 'model errors' includes errors in the initial and boundary conditions, as well as errors in the governing equations themselves). For most practical purposes computational methods must be used. When the governing equations are linear (e.g. the linearized quasi-geostrophic equations) a representer analysis can be used to reduce the dimension of the state space over which the minimizer is sought, from the number of degrees of freedom of the discretized equations, to the number of data values [3]. When the governing equations are nonlinear, however, this reduction of dimension can be carried out only by approximating the nonlinear EL equations with a sequence of linear equations and solving at each iteration (see [13] and [8] for applications to the two-dimensional Euler equations and the quasi-geostrophic equations). The validity of such iterative approaches depends upon the existence and regularity of the solutions to the nonlinear EL equations that the linear sequence of equations approximates.

This paper proves the existence of strong solutions of the generalized inverse of the quasi-geostrophic equations. The applicability of the result is therefore to provide confidence in the numerical iterative approaches described above and which are currently being explored as viable assimilation tools in the study of the atmosphere and oceans. Since the results presented below are valid on finite time intervals that depend on the norms of the initial conditions, they also provide information on the time intervals over which such numerical schemes can be expected to converge.

In particular, this paper proves the existence of solutions to the EL equations obtained by seeking a best fit to the quasi-geostrophic equations (dissipative and nondissipative) and to the data. The best fit is defined as the minimizer, belonging to a class of admissable functions, of a penalty functional that is a weighted sum of residuals representing errors in the equations, in the boundary and initial conditions, and in the data. The EL equations are then a coupled system comprising the quasi-geostrophic equations and their adjoint equations in a suitable Hilbert space formulation, as described in section 2. After some preliminaries in section 3, existence is proved for the nondissipative case in section 4 and for the dissipative case in section 5. The proof is shown to be valid on a finite-time interval that depends upon the initial conditions; because of the coupled nature of the EL equations an existence proof on an arbitrary, finite-time interval is not possible by the methods used here. The approach follows that of [14], used to prove the existence of solutions to the generalized inverse of the two-dimensional Euler equations, and makes use of the existence results of the quasi-geostrophic equations in [4] and [7]. An error in [14] is corrected and its effect on the time interval over which the proof is valid is indicated. Some remarks are given in section 6 on implications and extensions, as well as on the dependence of the time interval on the initial conditions and on the dissipation coefficient.

2. Model formulation

The quasi-geostrophic system used here is the same as that described in [4, 7], and derived fully in, e.g., [17], with constant horizontal and thermal dissipation coefficient, K (see [6, 17] for a discussion of this assumption), and zero vertical dissipation, describing the evolution of vorticity, ω and potential temperature, θ:

$$\omega_t + \boldsymbol{u}\cdot\nabla\omega - K\Delta\omega = \beta v + S_0 + s \qquad \text{in} \quad B_T, \tag{2.1a}$$

$$\omega = \omega^I + \varepsilon \qquad \text{in} \quad B \quad \text{at} \quad t = 0, \tag{2.1b}$$

$$\theta_t + \boldsymbol{u}\cdot\nabla\theta - K\Delta\theta = H_0^X + h^X \qquad \text{in} \quad \Omega_T^X, \tag{2.1c}$$

$$\theta = \theta^{XI} + \delta^X \qquad \text{in} \quad \Omega^X \quad \text{at} \quad t = 0, \tag{2.1d}$$

where $\boldsymbol{u} = (u, v)$ and (ω, θ) are related, at each time t, by an elliptic equation for the quasi-geostrophic streamfunction, ψ:

$$\tilde{\Delta}\psi = \Delta\psi + \rho^{-1}(\rho\alpha\psi_z)_z = \omega \qquad \text{in} \quad B, \tag{2.2a}$$

$$\psi_z = \theta^X \quad \text{in} \quad \Omega^X, \tag{2.2b}$$

$$\boldsymbol{u} = \nabla^\perp\psi \qquad \text{in} \quad B. \tag{2.2c}$$

The following notation has been used: $B = \Omega\times(0, h)$, where $\Omega = (0, 1)\times(0, 1)$, is the spatial domain, a rectangular box with coordinates $\boldsymbol{x} = (x, y, z) = (\boldsymbol{x}_H, z)$; $\Omega^z = (0, 1)\times(0, 1)\times z$ is a horizontal cross section at fixed z; $B_T = B\times(0, T)$, $\Omega_T = \Omega\times(0, T)$, $\Omega_T^z = \Omega^z\times(0, T)$ where $(0, T)$ is a fixed time interval; $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, $\nabla^\perp = (-\frac{\partial}{\partial y}, \frac{\partial}{\partial x})$, and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ are horizontal gradient and diffusion operators; subscripts x, y, z, t denote partial derivatives; S_0, H_0^X and ω^I, θ^{XI} are prescribed forcing functions and initial conditions; s, h^X and ε, δ^X are *undetermined* errors in the forcing and initial conditions; α and ρ are given, positive, increasing functions and β is a constant Coriolis parameter; the superscript $X = 0, h$ represents the lower or upper surface at $z = 0$ and $z = h$. All variables are considered to be periodic with period one in both horizontal directions. As noted in [7] embedding and completeness results for more general function spaces also hold for the restricted spaces of doubly periodic functions used here.

The nondissipative form of (2.1a)–(2.1d) and (2.2a)–(2.2c) with $K \equiv 0$, will be denoted by QG in the following, and the dissipative form, with $K > 0$, will be denoted by DQG. Also, the simplified QG or QGD equations (SQG or SQGD respectively) are obtained from (2.1a)–(2.1d) and (2.2a)–(2.2c) by setting $\theta^{XI} = \delta^X = H_0^X = h^X = 0$ in (2.1c) and (2.1d), so that $\theta \equiv 0$ on $\bar{\Omega}^X$, $X = 0, h, \forall t \in [0, T]$.

Setting s, h^X and ε, δ^X identically zero corresponds to the assumption that the model equations represent the dynamics perfectly and that the forcing functions and initial conditions are known exactly. Finite-time existence results for these 'forward' equations were presented in [4] ($K > 0$) and [7] ($K = 0$). When additional data is included, for example by requiring one of the model fields to be close to some measured value at a particular time and space location, the forward problem as stated above will in general become overspecified. For this reason the undetermined error terms s, h^X and ε, δ^X are included in the model equations. In practical applications these errors could arise from inaccuracies in the model equations (e.g. because of oversimplification of the full equations of motion) or from poor knowledge of actual forcing terms (e.g. heating) in the atmosphere or oceans.

The inverse problem consists of determining the error terms s, h^X, ε, δ^X given the additional data that is to be fitted by the model variables (ω, θ). Assuming N data measurements d_i, $i = 1\ldots N$, with unknown data error ϵ_i, the model variables can be related to the data through N measurement functionals:

$$\mathcal{L}_i(\omega, \theta) = d_i + \epsilon_i \qquad \text{in} \quad B_T \quad \text{for} \quad i = 1\ldots N. \tag{2.3}$$

Here $\mathcal{L}_i$ are linear measurement functionals of the form

$$\begin{aligned}\mathcal{L}_i(\omega,\theta) &= \int_{\bar{B}_T} \mathcal{K}_i G(\omega,\theta)\\ &= \int_{\bar{B}_T} \mathcal{K}_i \psi\\ &= \int_{\bar{B}_T} \mathcal{K}_i(\boldsymbol{x},t,\boldsymbol{x}_i,t_i)\psi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \qquad (2.4)\end{aligned}$$

where $\mathcal{K}_i(\cdot,\cdot,\boldsymbol{x}_i,t_i) : B_T \to R$ is any smooth function with support in a neighbourhood of the data location $(\boldsymbol{x}_i, t_i)$ [3]. Note that, at each t, the streamfunction ψ is determined uniquely by $(\omega, \theta_0, \theta_h)$ in (2.2a) through the Green's function representation

$$\psi = G(\omega,\theta) = \int_{\bar{B}} g\omega + \int_{\bar{\Omega}^0} g\theta_0 + \int_{\bar{\Omega}^h} g\theta_h, \qquad (2.5)$$

where g is the fundamental solution for $\tilde{\Delta}$. The role of $\mathcal{K}_i$ is to mimic the tendency of an actual measurement to represent the measured field over a finite area in space and time (i.e. not at an isolated point). That the streamfunction is used in (2.4) is for notational convenience only; the measurements could equally well be made on ω or θ or indeed on any derived quantity or combination thereof. In the following, vector notation will also be used for the measurement functionals, data and data error: $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \ldots, \mathcal{L}_N)$, etc.

The generalized inverse solution is defined as the minimizer of s, h^X, ε, δ^X, and $\boldsymbol{\epsilon}$ in a weighted, least-squares sense [3, 18]. In particular, it is the minimizer of the penalty functional $\mathcal{J}$ given by

$$\begin{aligned}\mathcal{J}[\omega,\theta] = &\int_{\bar{B}_T}\int_{\bar{B}_T} sWs + \int_{\bar{B}}\int_{\bar{B}} \varepsilon V \varepsilon\\ &+ \sum_{X=0,h}\left(\int_{\bar{\Omega}_T^X}\int_{\bar{\Omega}_T^X} h^X \hat{W} h^X + \int_{\bar{\Omega}^X}\int_{\bar{\Omega}^X} \delta^X \hat{V} \delta^X\right) + \sum_k\sum_l \epsilon_k w_{kl} \epsilon_l, \qquad (2.6)\end{aligned}$$

where the weighting kernels W, $\hat{W}$, V and $\hat{V}$ are real-valued functions of $B_T \times B_T$, $\Omega_T \times \Omega_T$, $B \times B$, $\Omega \times \Omega$ respectively and where $\boldsymbol{w} = [w_{kl}]$ is a $N \times N$ matrix where N is the number of data points. If the model errors s, h^X, ε and δ^X are considered as zero-mean random variables then the weighting kernels W, $\hat{W}$, V, and $\hat{V}$ are defined as the functional inverses of the auto-covariances Q, $\hat{Q}$, A and $\hat{A}$ of s, h^X, ε and δ^X [20]:

$$\int_{\bar{B}_T} Q(\boldsymbol{x},t,\boldsymbol{x}',t')W(\boldsymbol{x}',t',\boldsymbol{x}'',t'')\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}t' = \delta(\boldsymbol{x}-\boldsymbol{x}'', t-t''), \qquad (2.7)$$

and similar expressions for the other kernels. Similarly, $\boldsymbol{w}$ is taken as the matrix inverse of the covariance of the data error, also considered as a zero-mean random variable: $\boldsymbol{w} = \overline{\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T}$. In practice, the covariances Q, $\hat{Q}$, A, $\hat{A}$, and $\boldsymbol{w}^{-1}$ are usually poorly known, and one of the major challenges in modern data assimilation lies in their estimation. In the following the error terms s, h^X, ε, δ^X and $\boldsymbol{\epsilon}$ are considered rather as deterministic control variables, and the covariance kernels Q, etc are formally related to the weighting kernels W, etc, through expressions of type (2.7).

By the calculus of variations (e.g. [10]), any minimizer of (2.6) must satisfy the following coupled system, referred to as the EL equations:

$$\omega_t + \boldsymbol{u}\cdot\nabla\omega - K\Delta\omega = S^{(\omega)} \quad \text{in} \quad B_T, \qquad (2.8a)$$

$$\omega = \tilde{\omega}^I \quad \text{in} \quad B \quad \text{at} \quad t=0, \qquad (2.8b)$$

$$\theta_t + \boldsymbol{u}\cdot\nabla\theta - K\Delta\theta = H^{(\theta)} \quad \text{in} \quad \Omega_T^X, \qquad (2.8c)$$

$$\theta = \tilde{\theta}^I \quad \text{in} \quad \Omega^X \quad \text{at} \quad t=0, \qquad (2.8d)$$

$$-\mu_t - \boldsymbol{u} \cdot \nabla\mu - K\Delta\mu = S^{(\mu)} \quad \text{in} \quad B_T, \tag{2.9a}$$

$$\mu = 0 \quad \text{in} \quad B \quad \text{at} \quad t = T, \tag{2.9b}$$

$$-\lambda_t - \boldsymbol{u} \cdot \nabla\lambda - K\Delta\mu = H^{(\lambda)} \quad \text{in} \quad \Omega_T^X, \tag{2.9c}$$

$$\lambda = 0 \quad \text{in} \quad \Omega^X \quad \text{at} \quad t = T, \tag{2.9d}$$

for $X = 0, h$. Here

$$S^{(\omega)} = \beta v + S_0 + \int_{\bar{B}_T} Q\mu, \qquad \tilde{\omega}^I = \omega^I + \int_{\bar{B}} A\mu, \tag{2.10a}$$

$$H^{(\theta)} = H_0^X + \int_{\bar{\Omega}_T^X} \hat{Q}\lambda, \qquad \tilde{\theta}^I = \theta^{XI} + \int_{\bar{\Omega}^X} \hat{A}\lambda, \tag{2.10b}$$

$$S^{(\mu)} = G(\nabla^\perp\mu \cdot \nabla\omega + \beta\mu_x, \nabla^\perp\lambda \cdot \nabla\theta) + \mathcal{L}'(\delta_{B_T}, 0)\boldsymbol{D}[\boldsymbol{d} - \mathcal{L}(\omega, \theta)], \tag{2.10c}$$

$$H^{(\lambda)} = G(\nabla^\perp\mu \cdot \nabla\omega + \beta\mu_x, \nabla^\perp\lambda \cdot \nabla\theta) + \mathcal{L}'(0, \delta_{\Omega_T^X})\boldsymbol{D}[\boldsymbol{d} - \mathcal{L}(\omega, \theta)], \tag{2.10d}$$

where $G(\cdot, \cdot)$ is defined in (2.5), where $\delta_{B_T} = \delta_{B_T}(\boldsymbol{x}-\boldsymbol{x}', t-t')$ and $\delta_{\Omega_T^X} = \delta_{\Omega_T^X}(\boldsymbol{x}_H-\boldsymbol{x}'_H, t-t')$ are the Dirac delta functions on B_T and Ω_T^X, $X = 0, h$, and where the prime on $\mathcal{L}'$ indicates measurement on the primed arguments.

Equations (2.8*a*)–(2.8*d*) and (2.9*a*)–(2.9*d*) are called the forward and adjoint equations, respectively. Together they comprise a coupled system of nonlinear, first-order hyperbolic ($K = 0$) or parabolic ($K > 0$) equations with initial and final conditions specified. The proof of the existence of solutions, given in section 4 for the nondissipative case $K = 0$ and in section 5 for the dissipative case $K > 0$, is in two stages and follows closely that of [14]. First, suppose that (ω, θ) are given in a suitably defined function space $\mathcal{E}^{(\omega,\theta)}$ and suppose that (μ, λ) are given in a suitably defined function space $\mathcal{E}^{(\mu,\lambda)}$. By integrating along characteristics in the nondissipative case, and by considering heat potentials in the dissipative case, it is shown that the adjoint equations (2.9*a*)–(2.9*d*) define a map, $M^\dagger$, from $\mathcal{E}^{(\mu,\lambda)}$ to itself, by $(\mu', \lambda') = M^\dagger(\mu, \lambda)$, with

$$-\mu'_t - \boldsymbol{u}(\omega) \cdot \nabla\mu' - K\Delta\mu' = S^{(\mu)}(\omega, \theta, \mu, \lambda) \quad \text{in} \quad B_T \tag{2.11}$$

and a similar expression for λ', subject to the final conditions, $\mu' = \lambda' = 0$ at $t = T$. Since $M^\dagger$ is continuous and since $\mathcal{E}^{(\mu,\lambda)}$ is compact and convex, the Schauder fixed-point theorem, (e.g. [12, theorem 11.1]) implies there is a fixed point of $M^\dagger$, (μ^*, λ^*), for which $M^\dagger(\mu^*, \lambda^*) = (\mu^*, \lambda^*)$. The fixed point is a solution of (2.9*a*)–(2.9*d*) for given (ω, θ) in $\mathcal{E}^{(\omega,\theta)}$.

By a similar procedure, the forward equations (2.8*a*)–(2.8*d*), for the above (μ^*, λ^*) in $\mathcal{E}^{(\mu,\lambda)}$, define a map M from $\mathcal{E}^{(\omega,\theta)}$ to itself by $(\omega', \theta') = M(\omega, \theta)$. Again the Schauder fixed-point theorem implies there is a fixed point satisfying $M(\omega^*, \theta^*) = (\omega^*, \theta^*)$ that is a solution of (2.8*a*)–(2.8*d*). The functions $(\omega^*, \theta^*, \mu^*, \lambda^*)$ then satisfy the coupled EL equations (2.8*a*)–(2.8*d*) and (2.9*a*)–(2.9*d*) and comprise the desired solution. Further, by the construction of the spaces $\mathcal{E}^{(\omega,\theta)}$ and $\mathcal{E}^{(\mu,\lambda)}$, they have the required continuity and are strong solutions of the EL equations.

3. Function spaces

For completeness, the main function spaces used throughout the remainder of this paper are outlined here. Let $C^k(\bar{D})$ and be the usual space of kth-order continuous functions on a domain D, with associated norm $|f|_{k;\bar{D}} = \max_{|\alpha|\leqslant k} \sup_{x\in D} |D^\alpha f|$. Further, let $C^{k,l}(\bar{D}_T)$ be the space of functions on $D_T = D \times (0, T)$ that are in both $C^k(\bar{D})$ and $C^k([0, T])$ and define the associated norm by $\|f\|_{k,l;\bar{D}_T} = \max(|f|_{k;\bar{D}}, |f|_{l;[0,T]})$.

The function, f, is Hölder continuous with coefficient ν, denoted by $f \in C^{\nu}(\bar{D})$, if

$$[f]_\nu = \sup_{x,y\in D, x\neq y} |x-y|^{-\nu}|f(x)-f(y)| < C. \tag{3.1}$$

Then $C^{k+\nu}(\bar{D})$ denotes the subspace of $C^k(\bar{D})$ consisting of functions whose spatial partial derivatives $D^\alpha f$ of order $|\alpha| = k$ are Hölder continuous with coefficient ν. For anisotropic Hölder continuity in space and time $C^{k+\nu,l+\delta}(\bar{D}_T)$ denotes the subset of $C^{k,l}(\bar{D}_T)$ consisting of functions whose partial derivatives $D^\alpha f$ of order $|\alpha| = k$ are Hölder continuous with coefficient ν, uniformly in time, and whose temporal partial derivatives $\partial_t^l f$ of order l are Hölder continuous with coefficient δ, uniformly in space. The spaces $C^{k+\nu}(\bar{D})$ and $C^{k+\nu,l+\delta}(\bar{D}_T)$ are Banach spaces when equipped with the following norms [1]:

$$|f|_{k+\nu;\bar{D}} = |f|_{k;\bar{D}} + \max_{|\alpha|=k}[D^\alpha f]_\nu \tag{3.2}$$

$$\|f\|_{k+\nu,l+\delta;\bar{D}_T} = \|f\|_{k,l;\bar{D}_T} + \max_{|\alpha|=k}[D^\alpha f]_\nu + [\partial_t^l f]_\delta. \tag{3.3}$$

Finally, the function g satisfies a parabolic Hölder condition if

$$[g]_\nu^* = \sup_{(x,t)\neq(x',t')} ((x-x')\cdot(x-x') + |t-t'|)^{-\frac{\nu}{2}}|g(x,t)-g(x',t')| < C. \tag{3.4}$$

Then $C^{k+\nu,\frac{k+\nu}{2}}(\bar{D}_T)$ denotes the subspace of $C^{k,k/2}(\bar{D}_T)$ consisting of functions whose spatial partial derivatives $D^\alpha u$ of order $|\alpha| = k$ and whose temporal partial derivatives of order $k/2$ satisfy a parabolic Hölder condition with coefficient ν. The space $C^{k+\nu,\frac{k+\nu}{2}}(\bar{D}_T)$ is a Banach space when equipped with the following norm [16]:

$$[\![g]\!]_{k+\nu;\bar{D}_T} = \|g\|_{k,k/2;\bar{D}_T} + \max_{|\alpha|=k}[D^\alpha g]_\nu^* + [\partial_t^{k/2} f]_\nu^*. \tag{3.5}$$

When the norms are applied to vector valued functions, the max of the norms of the scalar components is taken. Throughout the following, generic constants will be denoted by c and specific, but undetermined, constants by c_1, c_2, etc.

4. Existence of the generalized inverse to the nondissipative equations

In this section the existence of solutions to (2.8*a*)–(2.8*d*) and (2.9*a*)–(2.9*d*) is proved in the special case $K = 0$. Under this assuption the forward and adjoint equations take a hyperbolic form and are treated by integrating along characteristics, in this case equivalent to the geostrophic particle paths. Since θ occurs in the equations in an identical form on the lower and upper boundaries, attention will be restricted to the case where θ is nonzero only on the lower boundary $\bar{\Omega}^0$, without any loss of generality. In addition, the following are assumed to hold throughout this section:

$$\alpha, \rho \in C^{1+\nu}([0,h]), \tag{4.1a}$$

$$Q \in C^{1+\nu,0}(\bar{B}_T) \times C^{1+\nu,0}(\bar{B}_T), \qquad A \in C^{1+\nu}(\bar{B}) \times C^{1+\nu}(\bar{B}), \tag{4.1b}$$

$$\hat{Q} \in C^{2+\nu,0}(\bar{\Omega}_T) \times C^{2+\nu,0}(\bar{\Omega}_T), \qquad \hat{A} \in C^{2+\nu}(\bar{\Omega}) \times C^{2+\nu}(\bar{\Omega}), \tag{4.1c}$$

$$\mathcal{K}_i \in C^{\nu,0}(\bar{B}_T) \times C^{\nu,0}(\bar{B}_T), \tag{4.1d}$$

$$\omega^I \in C^{1+\nu}(\bar{B}), \qquad \theta^I \in C^{2+\nu}(\bar{\Omega}), \tag{4.1e}$$

$$S_0 \in C^{1+\nu,0}(\bar{B}_T), \qquad H_0 \in C^{2+\nu,0}(\bar{\Omega}_T), \tag{4.1f}$$

and the constants $q, a, \hat{q}, \hat{a}$ and $K^I, K^S, \hat{K}^I, \hat{K}^S$ are defined by

$$q = \operatorname{diam}(\bar{B}) \sup_{\bar{B}_T} \|Q(\cdot,\cdot;\boldsymbol{x},t)\|_{1+\nu,0;\bar{B}_T}, \qquad a = \operatorname{diam}(\bar{B}) \sup_{\bar{B}} |A(\cdot;\boldsymbol{x})|_{1+\nu;\bar{B}}, \tag{4.2a}$$

$$\hat{q} = \mathrm{diam}(\bar{\Omega}) \sup_{\bar{\Omega}_T} \|\hat{Q}(\cdot,\cdot;\boldsymbol{x}_H,t)\|_{2+\nu,0;\bar{\Omega}_T}, \qquad \hat{a} = \mathrm{diam}(\bar{\Omega}) \sup_{\bar{\Omega}} |\hat{A}(\cdot;\boldsymbol{x}_H)|_{2+\nu;\bar{\Omega}}, \tag{4.2b}$$

$$K^I = |\omega^I|_{1+\nu;\bar{B}}, \qquad \hat{K}^I = |\theta^I|_{2+\nu;\bar{\Omega}}, \tag{4.2c}$$

$$K^S = \|S_0\|_{1+\nu,0;\bar{B}_T}, \qquad \hat{K}^S = \|H_0\|_{2+\nu,0;\bar{\Omega}_T}. \tag{4.2d}$$

Since $\tilde{\Delta}$ is strictly elliptic, modification of the standard results for Neumann boundary conditions to mixed Neumann, doubly periodic boundary conditions, as discussed in [7], implies that, for a given $(\omega,\theta) \in C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T)$ and for fixed t, (2.2*a*)–(2.2*c*) has a unique solution $\boldsymbol{u} \in C^{2+\nu}(\bar{B})$ that satisfies the Schauder estimate

$$|\boldsymbol{u}|_{2+\nu;\bar{B}} \leqslant c(|\omega|_{1+\nu;\bar{B}} + |\theta|_{2+\nu;\bar{\Omega}}). \tag{4.3}$$

Further, by linearity of the Green function representation of ψ, it follows that $\boldsymbol{u} \in C^{2+\nu,0}(\bar{B}_T)$ [7, equation 5.1ff], and, by taking suprema over t, the bound (4.3) can be extended to

$$\|\boldsymbol{u}\|_{2+\nu,0;\bar{B}_T} \leqslant c(\|\omega\|_{1+\nu,0;\bar{B}_T} + \|\theta\|_{2+\nu,0;\bar{\Omega}_T}). \tag{4.4}$$

Next define

$$\begin{aligned}\mathcal{E}^{(\mu,\lambda)} = \{&(\mu,\lambda) \in C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T) \bigcap C^{0,\delta_1}(\bar{B}_T) \\ &\times C^{0,\delta_1}(\bar{\Omega}_T) : \|\mu\|_{1+\nu,0;\bar{B}_T} \leqslant L^\mu \text{ and } \|\lambda\|_{2+\nu,0;\bar{\Omega}_T} \leqslant L^\lambda\},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathcal{E}^{(\omega,\theta)} = \{&(\omega,\theta) \in C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T) \bigcap C^{0,\delta_2}(\bar{B}_T) \\ &\times C^{0,\delta_2}(\bar{\Omega}_T) : \|\omega\|_{1+\nu,0;\bar{B}_T} \leqslant L^\omega \text{ and } \|\theta\|_{2+\nu,0;\bar{\Omega}_T} \leqslant L^\theta\},\end{aligned} \tag{4.6}$$

where L^μ, L^λ, L^ω, L^θ are positive constants and $0 < \delta_1, \delta_2 < 1$, and define $P = L^\mu + L^\lambda$ and $R = L^\omega + L^\theta$ so that $\omega \in \mathcal{E}^{(\omega)} \Rightarrow \|\boldsymbol{u}\|_{2+\nu,0;\bar{B}_T} \leqslant cR$, by (4.4). The velocity field $\boldsymbol{u}$ defines particle paths $\boldsymbol{X}_H = (X_1, X_2) = \boldsymbol{X}_H(\boldsymbol{x}_H, z, t, \tau)$ that are solutions to the following system of ordinary differential equations:

$$\frac{\mathrm{d}\boldsymbol{X}_H}{\mathrm{d}\tau} = \boldsymbol{u}(\boldsymbol{X}_H, z, \tau), \tag{4.7}$$

with initial/final time conditions $\boldsymbol{X}_H(\boldsymbol{x}_H, z, t, t) = \boldsymbol{x}_H$. The solution $\boldsymbol{X}_H$ to (4.7) describes the path of the fluid particle that is at position $(\boldsymbol{x}_H, z)$ at time $\tau = t$.

Lemma 4.1. *The particle paths, $\boldsymbol{X}_H(\boldsymbol{x}_H, z, t, \tau)$, defined by (4.7) are unique and satisfy $\boldsymbol{X}_H(\tau) \in C^{2+\nu,1}(\bar{B}_T)$ for each $\tau \in [0,T]$. Further, the following bound holds for any $t, \tau \in [0,T] \times [0,T]$:*

$$|\boldsymbol{X}_H(t,\tau)|_{2+\nu;\bar{B}} \leqslant \mathrm{e}^{c_1 R|t-\tau|}. \tag{4.8}$$

Proof. Continuity of $\boldsymbol{X}_H$ follows from standard results in ordinary differential equations (e.g. [15, theorem V3.1]) and the extension to Hölder continuity in space was described in [7]. (See also [7] for a discussion of the effect of double periodicity.) The bound (4.8) for arbitrary s, t is obtained by a straightforward extension of the arguments used in [7] to obtain bounds at $\tau = 0$. In particular, (4.8) is obtained by integrating the identities, $\frac{\mathrm{d}}{\mathrm{d}\tau} D\boldsymbol{X} = D\boldsymbol{u} D\boldsymbol{X}$, $\frac{\mathrm{d}}{\mathrm{d}\tau} D^2\boldsymbol{X} = D^2\boldsymbol{u}(D\boldsymbol{X})^2 + D\boldsymbol{u} D^2\boldsymbol{X}$, and a derived expression for $\frac{\mathrm{d}}{\mathrm{d}\tau}(D^2\boldsymbol{X}' - D^2\boldsymbol{X}'')$, where D and D^2 denote first- and second-order spatial derivatives. □

For a given $(\omega,\theta) \in \mathcal{E}^{(\omega,\theta)}$ the adjoint equations (2.9*a*)–(2.9*d*) define a map $M^\dagger$: $C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T) \to C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T)$ by $(\mu',\lambda') = M^\dagger(\mu,\lambda)$ where

$$-\mu'_t - \boldsymbol{u}(\omega,\theta) \cdot \nabla\mu' = S^{(\mu)}(\omega,\theta,\mu,\lambda) \qquad \text{in} \quad \bar{B}_T \tag{4.9}$$

$$-\lambda'_t - \boldsymbol{u}(\omega,\theta) \cdot \nabla\lambda' = H^{(\lambda)}(\omega,\theta,\mu,\lambda) \qquad \text{in} \quad \bar{\Omega}^0_T. \tag{4.10}$$

The particle paths are then used to integrate the adjoint equations (4.9) and (4.10) backwards in time from the final conditions, $(\mu', \lambda') = (0, 0)$ at time $t = T$. The formal solution is

$$\mu'(\boldsymbol{x}, t) = -\int_t^T \mathrm{d}\tau\, S^{(\mu)}(\boldsymbol{X}_H(\boldsymbol{x}_H, z, t, \tau), z, \tau) \tag{4.11}$$

$$\lambda'(\boldsymbol{x}_H, t) = -\int_t^T \mathrm{d}\tau\, H^{(\lambda)}(\boldsymbol{X}_H(\boldsymbol{x}_H, 0, t, \tau), 0, \tau). \tag{4.12}$$

To ensure $M^\dagger$ in fact maps $\mathcal{E}^{(\mu,\lambda)}$ to itself, continuity and bounds for the right-hand side of (4.9) and (4.10) are required, and are provided in the following two results.

Lemma 4.2. *Let* $(\omega, \theta) \in \mathcal{E}^{(\omega,\theta)}$ *and* $(\mu, \lambda) \in \mathcal{E}^{(\mu,\lambda)}$. *Then* $\mathcal{G} = G(\omega, \theta, \mu, \lambda) = G(\nabla^\perp\mu \cdot \nabla\omega + \beta\mu_x, \nabla^\perp\lambda \cdot \nabla\theta)$ *satisfies* $\mathcal{G} \in C^{2+\nu,0}(\bar{B}_T)$ *and* $\|\mathcal{G}\|_{2+\nu,0;\bar{B}_T} \leqslant c(L^\mu(L^\omega + \beta) + L^\lambda L^\theta)$.

Proof. For $\omega \in C^{1+\nu,0}(\bar{B}_T)$ and $\mu \in C^{1+\nu,0}(\bar{B}_T)$ the product $\nabla^\perp\mu \cdot \nabla\omega \in C^{\nu,0}(\bar{B}_T)$. Also, $\beta\mu_x \in C^{\nu,0}(\bar{B}_T)$. Define $f_1 = \nabla^\perp\mu \cdot \nabla\omega + \beta\mu_x$. Then $f_1 \in C^{\nu,0}(\bar{B}_T)$ and satisfies

$$\|f_1\|_{\nu,0;\bar{B}_T} \leqslant c(\|\nabla^\perp\mu\|_{\nu,0;\bar{B}_T}\|\nabla\omega\|_{\nu,0;\bar{B}_T} + \|\beta\mu_x\|_{\nu,0;\bar{B}_T}) \tag{4.13}$$

$$\leqslant c\|\mu\|_{1+\nu,0;\bar{B}_T}(\|\omega\|_{1+\nu,0;\bar{B}_T} + \beta). \tag{4.14}$$

Similarly, $f_2 = \nabla^\perp\lambda \cdot \nabla\theta \in C^{1+\nu,0}(\bar{\Omega}_T)$ and satisfies

$$\|f_2\|_{1+\nu,0;\bar{\Omega}_T} \leqslant c\|\lambda\|_{2+\nu,0;\bar{\Omega}_T}\|\theta\|_{2+\nu,0;\bar{\Omega}_T}. \tag{4.15}$$

Define $\phi = \mathcal{G}(f_1, f_2)$. Then, by arguments similar to those used to derive (4.4), $\phi \in C^{2+\nu,0}(\bar{B}_T)$ and satisfies $\|\phi\|_{2+\nu,0;\bar{B}_T} \leqslant c(\|f_1\|_{\nu,0;\bar{B}_T} + \|f_2\|_{1+\nu,0;\bar{\Omega}_T})$. By the definitions of the subspaces $\mathcal{E}^{(\omega,\theta)}$ and $\mathcal{E}^{(\mu,\lambda)}$ this is the desired bound. □

Lemma 4.3. *Let* $(\omega, \theta) \in \mathcal{E}^{(\omega,\theta)}$, *let* $\boldsymbol{\mathcal{L}}$ *be as defined in (2.4) and let* $\mathcal{K}_i(\cdot, \cdot, \boldsymbol{x}_i, t_i) \in C^{\nu,0}(\bar{B}_T)$ *be any Hölder continuous function with support in a neighbourhood of each data location* $(\boldsymbol{x}_i, t_i)$. *Define the constants* B_1 *and* B_2 *by*

$$B_1 = \max_{1\leqslant i,j\leqslant N} |D_{ij}| \max_{1\leqslant i\leqslant N} \|\mathcal{K}_i(\cdot, \cdot, \boldsymbol{x}_i, t_i)\|_{\nu,0;\bar{B}_T} \max_{1\leqslant j\leqslant N} |d_j| \tag{4.16}$$

$$B_2 = \max_{1\leqslant i,j\leqslant N} |D_{ij}| \max_{1\leqslant i\leqslant N} \|\mathcal{K}_i(\cdot, \cdot, \boldsymbol{x}_i, t_i)\|_{\nu,0;\bar{B}_T} \operatorname{diam}(B). \tag{4.17}$$

Then $\boldsymbol{\mathcal{L}}'(\delta_{B_T}, 0)\boldsymbol{D}[\boldsymbol{d} - \boldsymbol{\mathcal{L}}(\omega, \theta)] \in C^{2+\nu,0}(\bar{B}_T)$ *and satisfies*

$$\|\boldsymbol{\mathcal{L}}'(\delta_{B_T}, 0)\boldsymbol{D}[\boldsymbol{d} - \boldsymbol{\mathcal{L}}(\omega, \theta)]\|_{2+\nu,0;\bar{B}_T} \leqslant c(B_1 + B_2 T R), \tag{4.18}$$

where $R = L^\omega + L^\theta$ *as before. Also* $\boldsymbol{\mathcal{L}}'(0, \delta_{\Omega_T})\boldsymbol{D}[\boldsymbol{d} - \boldsymbol{\mathcal{L}}(\omega, \theta)] \in C^{2+\nu,0}(\bar{\Omega}_T)$ *and its norm with respect to* $C^{2+\nu,0}(\bar{\Omega}_T)$ *satisfies the same bound.*

Proof. Consider $\mathcal{L}_i'(\delta_{B_T}, 0) = \mathcal{L}_i'(\delta_{B_T}(\boldsymbol{x} - \boldsymbol{x}', t - t'), 0)$ with the definition (2.4) of $\boldsymbol{\mathcal{L}}$:

$$\mathcal{L}_i'(\delta_{B_T}, 0) = \int_{\bar{B}_T'} \mathcal{K}_i(\boldsymbol{x}', t', \boldsymbol{x}_i, t_i) \int_{\bar{B}''} g(\boldsymbol{x}', \boldsymbol{x}'')\delta_{B_T}(\boldsymbol{x} - \boldsymbol{x}'', t - t') \tag{4.19}$$

$$= \int_{\bar{B}'} \mathcal{K}_i(\boldsymbol{x}', t, \boldsymbol{x}_i, t_i) g(\boldsymbol{x}', \boldsymbol{x}) \tag{4.20}$$

$$= G(\mathcal{K}_i(\cdot, \cdot, \boldsymbol{x}_i, t_i), 0) \qquad \text{in} \quad \bar{B}_T, \tag{4.21}$$

where the primes on the domains of integration indicate integration over the correspondingly primed variable. Similarly,

$$\mathcal{L}_i'(0, \delta_{\Omega_T}) = G(\mathcal{K}_i(\cdot, \cdot, \boldsymbol{x}_i, t_i), 0) \quad \text{on} \quad \bar{\Omega}_T^0. \tag{4.22}$$

Thus, by the arguments similar to those used to derive (4.4), $\mathcal{L}_i'(\delta_{B_T}, 0) \in C^{2+\nu,0}(\bar{B}_T)$ and $\mathcal{L}_i'(0, \delta_{\Omega_T}) \in C^{2+\nu,0}(\bar{\Omega}_T)$, and satisfy the bound

$$\|\mathcal{L}_i'(\delta_{B_T}, 0)\|_{2+\nu,0;\bar{B}_T} \leqslant c \max_{1\leqslant i\leqslant N} \|\mathcal{K}(\cdot, \cdot, \boldsymbol{x}_i, t_i)\|_{\nu,0;\bar{B}_T}, \tag{4.23}$$

and the same bound for $\|\mathcal{L}_i'(0, \delta_{\Omega_T})\|_{2+\nu,0;\bar{\Omega}_T}$, whenever $\mathcal{K}_i(\cdot, \cdot, \boldsymbol{x}_i, t_i) \in C^{\nu,0}(\bar{B}_T)$.

Also,

$$\mathcal{L}_i(\omega, \theta) = \int_{\bar{B}_T} \mathcal{K}_i(\boldsymbol{x}, t, \boldsymbol{x}_i, t_i) G(\omega, \theta)(\boldsymbol{x}, t), \tag{4.24}$$

$$= \int_{\bar{B}_T} \mathcal{K}_i(\boldsymbol{x}, t, \boldsymbol{x}_i, t_i) \psi(\boldsymbol{x}, t), \tag{4.25}$$

so that, by (4.3) and the hypothesis that $(\omega, \theta) \in \mathcal{E}^{(\omega,\theta)}$,

$$|\mathcal{L}_i(\omega, \theta)| \leqslant cT \operatorname{diam}(B) \max_{1\leqslant i\leqslant N} \|\mathcal{K}(\cdot, \cdot, \boldsymbol{x}_i, t_i)\|_{\nu,0;\bar{B}_T} (L^\omega + L^\theta). \tag{4.26}$$

Using

$$\begin{aligned}\|\boldsymbol{\mathcal{L}}'(\delta_{B_T}, 0)\boldsymbol{D}[\boldsymbol{d} - \boldsymbol{\mathcal{L}}(\omega, \theta)]\|_{2+\nu,0;\bar{B}_T} \\ \leqslant N^2 \max_{1\leqslant i,j\leqslant N} |D_{ij}| \|\mathcal{L}_i'(\delta_{B_T}, 0)\|_{2+\nu,0;\bar{B}_T} |\boldsymbol{d} - \boldsymbol{\mathcal{L}}(\omega, \theta)|,\end{aligned} \tag{4.27}$$

and a similar expression for $\|\boldsymbol{\mathcal{L}}'(0, \delta_{\Omega_T})\boldsymbol{D}[\boldsymbol{d} - \boldsymbol{\mathcal{L}}(\omega, \theta)]\|_{2+\nu,0;\bar{\Omega}_T}$, and substituting (4.23) and (4.26), the desired bounds follow. The continuity follows immediately from the continuity of $\mathcal{L}_i'(\delta_{B_T}, 0)$ and $\mathcal{L}_i'(0, \delta_{\Omega_T})$. □

Lemmas 4.2 and 4.3 together ensure the continuity and boundedness of $S^{(\mu)}$ and $H^{(\lambda)}$. Having established these it is now possible to show that, under suitable conditions, $M^\dagger$ maps $\mathcal{E}^{(\mu,\lambda)}$ into $\mathcal{E}^{(\mu,\lambda)}$.

Lemma 4.4. *Given $(\omega, \theta) \in \mathcal{E}^{(\omega,\theta)}$ and $(\mu, \lambda) \in \mathcal{E}^{(\mu,\lambda)}$, $M^\dagger$ maps $\mathcal{E}^{(\mu,\lambda)}$ into $\mathcal{E}^{(\mu,\lambda)}$ provided the following hold:*

$$T < \frac{1}{c_1 R} \log\left(\frac{c_1 R}{2(R+\beta)} + 1\right), \tag{4.28}$$

$$L^\mu, L^\lambda \geqslant \frac{(B_1 + B_2 T R)(\mathrm{e}^{c_1 R T} - 1)}{c_1 R - 2(R+\beta)(\mathrm{e}^{c_1 R T} - 1)}. \tag{4.29}$$

Proof. The $C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T)$ continuity of (μ', λ') follows from the continuity of the right-hand sides of (4.11) and (4.12). Bounds on $\|\mu'\|_{1+\nu,0;\bar{B}_T}$ and $\|\lambda'\|_{2+\nu,0;\bar{\Omega}_T}$ follow from (4.11) and (4.12) together with lemmas 4.1–4.3, the chain rule, and the bound (4.8) for $\|\boldsymbol{X}_H\|_{2+\nu,0;\bar{B}_T}$. In particular, (4.11) with the chain rule gives the bound

$$|\mu'(t)|_{1+\nu;\bar{B}} \leqslant \|S^{(\mu)}\|_{1+\nu,0;\bar{B}_T} \int_t^T \mathrm{d}\tau |\boldsymbol{X}_H(t, \tau)|_{1+\nu;\bar{B}}, \tag{4.30}$$

which gives, on substituting (4.8), integrating, and taking the supremum over t

$$\|\mu'\|_{1+\nu,0;\bar{B}_T} \leqslant \frac{1}{c_1 R}(\mathrm{e}^{c_1 R T} - 1)\|S^{(\mu)}\|_{1+\nu,0;\bar{B}_T}. \tag{4.31}$$

The bounds given by lemmas 4.2 and 4.3 then give

$$\|\mu'\|_{1+\nu,0;\bar{B}_T} \leqslant \frac{1}{c_1 R}(\mathrm{e}^{c_1 R T} - 1)(L^\mu(L^\omega + \beta) + L^\lambda L^\theta + B_1 + cT B_2 R). \tag{4.32}$$

A similar procedure establishes the same bound for $\|\lambda'\|_{2+\nu,0;\bar{\Omega}_T}$. Combining the two bounds and rearranging shows that $\|\mu'\|_{1+\nu,0;\bar{B}_T} \leqslant L^{\mu}$ and $\|\lambda'\|_{2+\nu,0;\bar{\Omega}_T} \leqslant L^{\lambda}$ provided the conditions (4.28) and (4.29) are satisfied†. The Hölder continuity in time of (μ', λ') follows from (4.11) and (4.12) and from the Hölder continuity in space and time of $\boldsymbol{X}_H$ by an argument similar to that described in [14]. □

The existence of a solution (μ^*, λ^*) to the adjoint equations (2.9*a*)–(2.9*d*) now follows by virtue of the existence of a fixed point of the map $M^{\dagger}$.

Lemma 4.5. *Suppose* $(\omega, \theta) \in \mathcal{E}^{(\omega,\theta)}$ *is given and that the conditions of lemma 4.4 are satisfied. Then there exists a solution* (μ^*, θ^*) *of (2.9a)–(2.9d) with* $(\mu^*, \theta^*) \in \mathcal{E}^{(\mu,\lambda)}$ *and with* $(\mu^*, \lambda^*) \in C^1([0, T])$.

Proof. First note that, since any linear combination of functions in $\mathcal{E}^{(\mu,\lambda)}$ is also in $\mathcal{E}^{(\mu,\lambda)}$, $\mathcal{E}^{(\mu,\lambda)}$ is a nonempty and convex subset of $C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T) \bigcap C^{0,\delta_1}(\bar{B}_T) \times C^{0,\delta_1}(\bar{\Omega}_T)$. Further, since $\mathcal{E}^{(\mu,\lambda)}$ consists of uniformly bounded functions that satisfy a uniform Hölder condition in both space and time as well as a uniform Hölder condition in space on their first spatial derivatives, these functions are equicontinuous in $\bar{B}_T \times \bar{\Omega}_T$. Therefore, by the Ascoli–Arzela theorem (e.g. [1, theorem 1.30]), $\mathcal{E}^{(\mu,\lambda)}$ is precompact and hence compact in $C^0(\bar{B}_T) \times C^0(\bar{\Omega}_T)$. Finally, $M^{\dagger}$ is linear from $\mathcal{E}^{(\mu,\lambda)}$ into $\mathcal{E}^{(\mu,\lambda)}$ and hence continuous in $C^0(\bar{B}_T) \times C^0(\bar{\Omega}_T)$. The result is then proved by applying Schauder's fixed-point theorem (e.g. [12]) to $M^{\dagger} : \mathcal{E}^{(\mu,\lambda)} \to \mathcal{E}^{(\mu,\lambda)}$, which implies a fixed point of $M^{\dagger}$ satisfying $M^{\dagger}\mu^* = \mu^*$. By the definition (4.11), (4.12) of $M^{\dagger}$, (μ^*, λ^*) is the required solution of (2.9*a*)–(2.9*d*). Finally, consideration of (4.9) and (4.10) shows that μ^*_t and λ^*_t are both $C^0([0, T])$ since all other terms are $C^0([0, T])$. □

The existence of a solution to the forward equations, (2.8*a*)–(2.8*d*), follows by similar arguments to those of the adjoint equations. For the solution $(\mu^*, \lambda^*) \in \mathcal{E}^{(\mu,\lambda)}$ given above, (2.8*a*)–(2.8*d*) defines a map $M : C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T) \to C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T)$ by $(\omega', \theta') = M(\omega, \theta)$ where

$$\omega'_t + \boldsymbol{u}(\omega, \theta) \cdot \nabla\omega' = S^{(\omega)}(v(\omega, \theta), \mu^*, \lambda^*) \qquad \text{in} \quad \bar{B}_T \tag{4.33}$$

$$\theta'_t + \boldsymbol{u}(\omega, \theta) \cdot \nabla\theta' = H^{(\theta)}(\mu^*, \lambda^*) \quad \text{in} \quad \bar{\Omega}^0_T, \tag{4.34}$$

with the initial conditions, $\omega' = \tilde{\omega}^I$, $\theta' = \tilde{\theta}^I$ at $t = 0$ and where $\tilde{\omega}^I$, $\tilde{\theta}^I$, $S^{(\omega)}$, $H^{(\theta)}$ are given by (2.10*a*), (2.10*b*). This time, the particle paths are used to integrate the equations (4.33) and (4.34) forwards in time, the formal solution being

$$\omega'(\boldsymbol{x}, t) = \omega^I(\boldsymbol{X}_H(\boldsymbol{x}_H, z, 0, t), z) + \int_0^t \mathrm{d}\tau\, S^{(\omega)}(\boldsymbol{X}_H(\boldsymbol{x}_H, z, t, \tau), z, \tau) \tag{4.35}$$

$$\theta'(\boldsymbol{x}_H, t) = \theta^I_0(\boldsymbol{X}_H(\boldsymbol{x}_H, 0, 0, t)) + \int_0^t \mathrm{d}\tau\, H^{(\theta)}(\boldsymbol{X}_H(\boldsymbol{x}_H, 0, t, \tau), 0, \tau). \tag{4.36}$$

As before, under suitable conditions, M maps $\mathcal{E}^{(\omega,\theta)}$ into $\mathcal{E}^{(\omega,\theta)}$:

† Note that in [14] the chain rule was never used in the derivation of the bounds on the adjoint variable or on the forward variable [14, equations (4.31) and (4.51)], and consequently no account was taken of the boundedness of the particle paths. Although the proof in [14] remains valid when this omission is corrected, the maximal time interval over which the proof is valid (obtained from [14, equation 4.55]) must be revised downward.

Lemma 4.6. *If the conditions of lemma 4.4 are satisfied then M maps $\mathcal{E}^{(\omega,\theta)}$ into $\mathcal{E}^{(\omega,\theta)}$ provided the following hold:*

$$L^{\omega} - K^{I}\mathrm{e}^{c_1RT} \geqslant \frac{(B_1 + B_2TR)(\mathrm{e}^{c_1RT} - 1)}{c_1R - 2(R+\beta)(\mathrm{e}^{c_1RT} - 1)}\left(a\mathrm{e}^{c_1RT} + \frac{qT^2}{c_1R}(\mathrm{e}^{c_1RT} - 1)\right) + \left(\beta + \frac{TK^S}{c_1R}\right)(\mathrm{e}^{c_1RT} - 1) \tag{4.37}$$

and

$$L^{\theta} - \hat{K}^{I}\mathrm{e}^{c_1RT} \geqslant \frac{(B_1 + B_2TR)(\mathrm{e}^{c_1RT} - 1)}{c_1R - 2(R+\beta)(\mathrm{e}^{c_1RT} - 1)}\left(\hat{a}\mathrm{e}^{c_1RT} + \frac{\hat{q}T^2}{c_1R}(\mathrm{e}^{c_1RT} - 1)\right) + \frac{T\hat{K}^S}{c_1R}(\mathrm{e}^{c_1RT} - 1). \tag{4.38}$$

Further, for any given K^I, $\hat{K}^I$, K^S and $\hat{K}^S$ there exist values of T, L^{ω} and L^{θ} that satisfy (4.37) and (4.38).

Proof. Recall that $S^{(\omega)}$ and $H^{(\theta)}$ are given by (2.10*a*) and (2.10*b*). Continuity of the terms on the right-hand sides of (4.35) and (4.36) follows immediately from the assumptions on the initial conditions and forcing terms given in (4.1*e*) and (4.1*f*), from the assumptions on the covariance kernels (4.1*b*) and (4.1*c*), and from the continuity of the particle paths $\boldsymbol{X}_H(\boldsymbol{x}_H, z, \tau, t)$ given in lemma 4.1. Bounds on terms involving the initial conditions and forcing terms are written using (4.8) and the definitions (4.2*c*) and (4.2*d*), and bounds on the terms involving the covariance kernels are written using (4.8) and straightforward expressions of the form $\|\int_{\bar{B}_T} Q\mu^*\|_{1+\nu,0;\bar{B}_T} \leqslant qT\|\mu^*\|_{0;\bar{B}_T}$, etc, using the definitions (4.2*a*) and (4.2*b*). As in lemma 4.4, the chain rule must be used. Note that the term involving βv in $S^{(\omega)}$ in (4.35) can be integrated directly to $\beta(y - X_2(\boldsymbol{x}_H, z, t, 0))$. Collecting all the terms, (4.35) and (4.36) finally yield

$$\|\omega'\|_{1+\nu,0;\bar{B}_T} \leqslant K^{I}\mathrm{e}^{c_1RT} + \left(\beta + \frac{TK^S}{c_1R}\right)(\mathrm{e}^{c_1RT} - 1) + \left(a\mathrm{e}^{c_1RT} + \frac{qT^2}{c_1R}(\mathrm{e}^{c_1RT} - 1)\right)L^{\mu} \tag{4.39}$$

$$\|\theta'\|_{2+\nu,0;\bar{\Omega}_T} \leqslant \hat{K}^{I}\mathrm{e}^{c_1RT} + \frac{T\hat{K}^S}{c_1R}(\mathrm{e}^{c_1RT} - 1) + \left(\hat{a}\mathrm{e}^{c_1RT} + \frac{\hat{q}T^2}{c_1R}(\mathrm{e}^{c_1RT} - 1)\right)L^{\lambda}, \tag{4.40}$$

where $(\|\mu^*\|_{0;\bar{B}_T}, \|\lambda^*\|_{0;\bar{\Omega}_T}) \leqslant (L^{\mu}, L^{\lambda})$ has also been used. As in lemma 4.4, combining the two bounds and rearranging shows that $\|\omega'\|_{1+\nu,0;\bar{B}_T} \leqslant L^{\omega}$ and $\|\theta'\|_{2+\nu,0;\bar{\Omega}_T} \leqslant L^{\theta}$ provided the conditions (4.37) and (4.38) are satisfied. The Hölder continuity in time of (ω', θ') follows from (4.35) and (4.36) and from the Hölder continuity in space and time of $\boldsymbol{X}_H$. Thus $(\omega', \theta') \in \mathcal{E}^{(\omega,\theta)}$ provided (4.37) and (4.38) are satisfied. Finally, since the right-hand side terms of (4.37) and (4.38) are increasing functions of both T and $R = L^{\omega} + L^{\theta}$ that are identically zero when $T = 0$, that is they reduce to $L^{\omega} \geqslant K^I$, $L^{\theta} \geqslant \hat{K}^I$ when $T = 0$, it follows that there exists $L^{\omega}, L^{\theta}, T > 0$ that satisfy (4.37) and (4.38) with $K^I < L^{\omega} < \infty$, $\hat{K}^I < L^{\theta} < \infty$, and $0 < T < \frac{1}{c_1R}\log(\frac{c_1R}{2(R+\beta)} + 1)$. □

Lemma 4.7. *The map M from $\mathcal{E}^{(\omega,\theta)}$ to itself, defined in (4.35) and (4.36) by $M(\omega, \theta) = (\omega', \theta')$, is continuous in the C^0 norm.*

Proof. Let $(\omega, \theta), (\omega_n, \theta_n) \in C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T)$ be such that $(\omega_n, \theta_n) \to (\omega, \theta)$ in the $C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T)$ norm, and define $(\omega_n', \theta_n') = M(\omega_n, \theta_n)$ for each n. Define ψ and

ψ_n to be the streamfunctions corresponding to (ω, θ) and (ω_n, θ_n). Then by the linearity of the boundary value problem, (2.2*a*)–(2.2*c*), and the arguments used to establish (4.4),

$$\|\psi_n - \psi\|_{2+\nu,0;\bar{B}_T} \leqslant c(\|\omega_n - \omega\|_{\nu,0;\bar{B}_T} + \|\theta_n - \theta\|_{1+\nu,0;\bar{\Omega}_T}). \tag{4.41}$$

Thus, $\psi_n \to \psi$ in $C^{2+\nu,0}(\bar{B}_T)$, and so the corresponding velocities, $\boldsymbol{u}_n \to \boldsymbol{u}$ in $C^{1+\nu,0}(\bar{B}_T)$, and the corresponding streamlines, $\boldsymbol{X}_{H_n} \to \boldsymbol{X}_H$ in $C^1(\bar{B}_T \times [0, T])$ [15, theorem V3.1]. Therefore, by the continuity of (μ^*, θ^*), (4.35) and (4.36) imply $(\omega_n', \theta_n') \to (\omega', \theta')$ in $C^0(\bar{B}_T)$. □

Theorem 4.1. *Under the assumptions (4.1a)–(4.1f) there is a time interval* $[0, T^*]$ *with* $0 < T^* < \frac{1}{c_1 R^I}\log(\frac{c_1 R^I}{2(R^I+\beta)} + 1)$, *where* $R^I = K^I + \hat{K}^I$, *such that there exists a strong solution* $(\omega^*, \theta^*, \mu^*, \lambda^*)$ *to (2.8a)–(2.8d) and (2.9a)–(2.9d) defined on the domain* $\bar{B} \times [0, T^*]$, *with* $(\omega^*, \theta^*) \in \mathcal{E}^{(\omega,\theta)}(T^*)$ *and* $(\mu^*, \lambda^*) \in \mathcal{E}^{(\mu,\lambda)}(T^*)$ *and with both* (ω^*, θ^*) *and* $(\mu^*, \lambda^*) \in C^1([0, T^*])$

Proof. Let $T^*(K^I, \hat{K}^I, K^S, \hat{K}^S)$ be the maximal T for which L^ω, L^θ can be found that satisfy (4.37) and (4.38), and which also satisfies (4.28) for such L^ω, L^θ. As in lemma 4.5 $\mathcal{E}^{(\omega,\theta)}$ is a nonempty, convex, and compact subset of $C^{1+\nu,0}(\bar{B}_T) \times C^{2+\nu,0}(\bar{\Omega}_T) \bigcap C^{0,\delta_1}(\bar{B}_T) \times C^{0,\delta_1}(\bar{\Omega}_T)$. By lemma 4.7, M is continuous in $C^0(\bar{B}_T) \times C^0(\bar{\Omega}_T)$. Again applying Schauder's fixed-point theorem to $M : \mathcal{E}^{(\omega,\theta)} \to \mathcal{E}^{(\omega,\theta)}$ implies there is a fixed point (ω^*, θ^*) of M, satisfying $M(\omega^*, \theta^*) = (\omega^*, \theta^*)$. Lemma 4.5 implies the existence and time regularity of the corresponding adjoint solution $(\mu^*, \lambda^*) \in \mathcal{E}_1$. By the definition (4.35) and (4.36) of M and the definition (4.11) and (4.12) of L, $(\omega^*, \theta^*, \mu^*, \lambda^*)$ is the required solution of (2.8*a*)–(2.8*d*) and (2.9*a*)–(2.9*d*). Finally, consideration of (4.33) and (4.34) shows that ω_t^* and θ_t^* are both $C^0([0, T^*])$ since all other terms are $C^0([0, T^*])$. □

5. Existence of the generalized inverse to the dissipative equations

5.1. Restriction to $\boldsymbol{\beta = 0}$, $\boldsymbol{\theta = 0}$

In this section the existence of solutions to (2.8*a*)–(2.8*d*) and (2.9*a*)–(2.9*d*) is proved for $K > 0$ under the simplifying assumptions that $\theta \equiv 0$ on $\bar{\Omega}^X$, $X = 0, h$ $\forall t \in [0, T]$ (SQGD) and that $\beta = 0$ (the f-plane). These assumptions are made for the sake of clarity; section 5.2 contains details of how the proof is extended when they are relaxed. Extensive use is made of the results of section 4 and of results from [4, 16]. Because of the different structure of the evolution equations when $K > 0$ (parabolic compared with hyperbolic) slightly different function spaces and assumptions are needed from those in section 4.

Let $C^{k+\nu}(\bar{D})$ and $|\cdot|_{k+\nu;\bar{D}}$ be the kth-order Hölder space and associated norm on a domain D and let $C^{k+\nu,\frac{k+\nu}{2}}(\bar{D}_T)$ and $\|\cdot\|_{k+\nu;\bar{D}_T}$ be the kth-order parabolic Hölder space and associated norm on $D_T = D \times (0, T)$, as defined in section 3. At times in this section, continuity in z will be treated separately from that in Ω_T^z, using the inequality $\|\cdot\|_{\nu;\bar{B}_T} \leqslant \sup_{[0,h]} \|\cdot\|_{\nu;\bar{\Omega}_T^z} + \sup_{\Omega_T} |\cdot|_{\nu;[0,h]}$. The following assumptions are made throughout:

$$\alpha, \rho \in C^{1+\nu}([0, h]), \tag{5.1a}$$

$$Q, \mathcal{K}_i \in C^{\nu,\frac{\nu}{2}}(\bar{B}_T) \times C^{\nu,\frac{\nu}{2}}(\bar{B}_T), \qquad A \in C^{2+\nu}(\bar{B}) \times C^{2+\nu}(\bar{B}), \tag{5.1b}$$

$$\omega^I, \nabla\omega^I, \Delta\omega^I \in C^\nu(\bar{B}), \qquad S_0 \in C^{\nu,\frac{\nu}{2}}(\bar{B}_T), \tag{5.1c}$$

and the constants q, a and K^I, K^S are defined by

$$q = \operatorname{diam}(\bar{B}) \sup_{\bar{B}_T} \|Q(\cdot, \cdot; \boldsymbol{x}, t)\|_{\nu;\bar{B}_T}, \qquad a = \operatorname{diam}(\bar{B}) \sup_{\bar{B}} |A(\cdot; \boldsymbol{x})|_{\nu;\bar{B}}, \tag{5.2a}$$

$$K^I = |\omega^I|_{\nu;\bar{B}} + |\nabla\omega^I|_{\nu;\bar{B}} + |\Delta\omega^I|_{\nu;\bar{B}}, \qquad K^S = \|S_0\|_{\nu;\bar{B}_T}. \tag{5.2b}$$

For a given $\omega \in C^{\nu}(\bar{B})$, (2.2$a$)–(2.2$c$) with $\theta \equiv 0$ has a unique solution $\boldsymbol{u} \in C^{1+\nu}(\bar{B})$ that satisfies the Schauder estimate [12]:

$$|\boldsymbol{u}|_{1+\nu;\bar{B}} \leqslant c|\omega|_{\nu;\bar{B}}. \tag{5.3}$$

As before, for a given $\omega \in C^{\nu,\frac{\nu}{2}}(\bar{B}_T)$, it follows by linearity of the Green function representation of ψ that $\boldsymbol{u} \in C^{1+\nu,\frac{1+\nu}{2}}(\bar{B}_T)$ and (5.3) can be extended to

$$\|\boldsymbol{u}\|_{1+\nu;\bar{B}_T} \leqslant c\|\omega\|_{\nu;\bar{B}_T}. \tag{5.4}$$

Define

$$\begin{aligned}\mathcal{E}^{(\mu)} = \{\mu \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T) : \|\mu\|_{2+\nu;\bar{\Omega}^z_T} \leqslant L^{\mu}_H, \forall z \in [0,h],\\ \text{with} \quad \mu, \nabla\mu \in C^{\nu}([0,h])\\ \text{and } |\mu|_{\nu;[0,h]}, |\nabla\mu|_{\nu;[0,h]} \leqslant L^{\mu}_{(z)}, \forall(\boldsymbol{x}_H,t) \in \bar{\Omega}_T\},\end{aligned} \tag{5.5}$$

$$\begin{aligned}\mathcal{E}^{(\omega)} = \{\omega \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T) : \|\omega\|_{2+\nu;\bar{\Omega}^z_T} \leqslant L^{\omega}_H, \forall z \in [0,h],\\ \text{with } \omega, \nabla\omega \in C^{\nu}([0,h])\\ \text{and } |\omega|_{\nu;[0,h]}, |\nabla\omega|_{\nu;[0,h]} \leqslant L^{\omega}_{(z)}, \forall(\boldsymbol{x}_H,t) \in \bar{\Omega}_T\},\end{aligned} \tag{5.6}$$

where $L^{\mu}_H, L^{\mu}_{(z)}, L^{\omega}_H, L^{\omega}_{(z)}$ are positive constants. Define $L^{\mu} = L^{\mu}_H + L^{\mu}_{(z)}$ and $L^{\omega} = L^{\omega}_H + L^{\omega}_{(z)}$ so that $\omega \in \mathcal{E}^{(\omega)} \Rightarrow \|\boldsymbol{u}\|_{1+\nu;\bar{B}_T} \leqslant cL^{\omega} = U$, by (5.4). Then, following [4], for a given $\omega \in \mathcal{E}^{(\omega)}$, and treating z as a parameter, (2.9a) and (2.9b) define a map $M^{\dagger} : C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T) \to C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T)$ by $\mu' = M^{\dagger}\mu$ with

$$\mu'(\boldsymbol{x}_H,z,t) = \int_t^T \mathrm{d}\tau \int_{\bar{\Omega}^z_T} \mathrm{d}\boldsymbol{x}'_H \Gamma^{\dagger}_U(\boldsymbol{x}_H,\boldsymbol{x}'_H,z,t,\tau) S^{(\mu)}(\boldsymbol{x}'_H,z,\tau), \tag{5.7}$$

where $S^{(\mu)}$ is given by (2.10c) and where $\Gamma^{\dagger}_U$ is the heat potential for the linear parabolic equation

$$-\mu'_t - \boldsymbol{u}(\omega)\cdot\nabla\mu' - K\Delta\mu' = S^{(\mu)}(\omega,\mu) \qquad \text{in} \quad \Omega^z_T. \tag{5.8}$$

Lemma 5.1. *Given $\omega \in \mathcal{E}^{(\omega)}$, $M^{\dagger}$ maps $\mathcal{E}^{(\mu)}$ into $\mathcal{E}^{(\mu)}$ provided the following hold:*

$$c_2 L^{\omega} < 1, \tag{5.9}$$

$$L^{\mu} \geqslant c_2 \frac{B_1 + B_2 T L^{\omega}}{1 - c_2 L^{\omega}}, \tag{5.10}$$

where $c_2 = c_2(T,U,K)$ is a constant depending on T, U and K such that c_2 is continuous and increasing in T with $c_2(0,U,K) = 0$. B_1 and B_2 are constants depending on the data parameters, N, D_{ij} and measurement functional, $\boldsymbol{\mathcal{L}}$, as described in lemma 4.3.

Proof. If $\omega \in \mathcal{E}^{(\omega)}$ and $\mu \in \mathcal{E}^{(\mu)}$ then by a simple extension of lemmas 4.2 and 4.3 (as for (5.4) above) $S^{(\mu)} \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{B}_T)$ and $\|S^{(\mu)}\|_{2+\nu;\bar{B}_T} \leqslant cL^{\mu}L^{\omega} + B_1 + B_2TL^{\omega}$. Also, $\omega \in \mathcal{E}^{(\omega)} \Rightarrow \|\boldsymbol{u}\|_{1+\nu;\bar{B}_T} \leqslant U$. Then by standard results from the theory of linear parabolic equations, $\mu' \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T)$ and satisfies the following *a priori* estimate

$$\|\mu'\|_{2+\nu;\bar{\Omega}^z_T} \leqslant c_2(T,U,K)\|S^{(\mu)}\|_{\nu;\bar{\Omega}^z_T}, \tag{5.11}$$

for every $z \in (0,h)$, where c_2 has the form $c_2 = \alpha_1 T^{\frac{\nu}{2}} + \alpha_2 T^{\frac{1+\nu}{2}} + \alpha_3 T$ ([16, Ch 4 section 14]). Thus $\|\mu'\|_{2+\nu;\bar{\Omega}^z_T} \leqslant L^{\mu}_H$ provided $c_2(L^{\mu}L^{\omega}+B_1+B_2TL^{\omega}) \leqslant L^{\mu}_H$. Following [4, equation 5.29ff] and writing $\delta\mu' = \mu'(z_2) - \mu'(z_1)$ and similarly for $\delta\boldsymbol{u}$, etc, $\delta\mu'$ satisfies

$$-\delta\mu'_t - \boldsymbol{u}|_{z_2}\cdot\nabla\delta\mu' - K\Delta\delta\mu' = \delta S^{(\mu)} - \delta\boldsymbol{u}\cdot\nabla\mu'|_{z_1} \qquad \text{in} \quad \Omega^{z_2}_T, \tag{5.12}$$

with $\delta\mu' = 0$ at $t = T$. Thus $\delta\mu' \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}_T^{z_2})$ and $\|\delta\mu'\|_{2+\nu;\bar{\Omega}_T^{z_2}} \leqslant c_2(\|\delta S^{(\mu)} + \delta\boldsymbol{u}\cdot\nabla\mu'\|_{\nu;\bar{\Omega}_T^{z_2}})$. Dividing by $|z_2 - z_1|^\nu$ and taking suprema over $[0, h]$ gives

$$|\mu'|_{\nu;[0,h]}, |\nabla\mu'|_{\nu;[0,h]}, |\Delta\mu'|_{\nu;[0,h]} \leqslant c_2(\|S^{(\mu)}\|_{\nu;\bar{B}_T} + \|\boldsymbol{u}\|_{\nu;\bar{B}_T} \sup_{z\in[0,h]} \|\nabla\mu'\|_{\nu;\bar{\Omega}_T^z}), \tag{5.13}$$

and, using (5.11) and the bound on $\|S^{(\mu)}\|_{\nu;\bar{B}_T}$, it follows that $|\mu'|_{\nu;[0,h]}, |\nabla\mu'|_{\nu;[0,h]} \leqslant L^\mu_{(z)}$ provided $c_2(L^\mu L^\omega + B_1 + B_2 T L^\omega) \leqslant L^\mu_{(z)}$. Combining the conditions for L^μ_H and $L^\mu_{(z)}$ and rescaling c_2 gives $\mu' \in \mathcal{E}^{(\mu)}$, provided (5.10) holds. □

Lemma 5.2. *Suppose $\omega \in \mathcal{E}^{(\omega)}$ is given and that the conditions of lemma 5.1 are satisfied. Then there exists a solution μ^* of (2.9a), (2.9b) with $\mu^* \in \mathcal{E}^{(\mu)}$.*

Proof. First note that $\mathcal{E}^{(\mu)}$ is a nonempty and convex subset of $C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}_T^z) \cap C^\nu([0, h])$. Further, since $\mathcal{E}^{(\mu)}$ contains functions that satisfy a Hölder condition in all variables, and hence are equicontinuous in B_T, the Ascoli–Arzela theorem (e.g. [1]) implies that $\mathcal{E}^{(\mu)}$ is compact. Finally, $M^\dagger$ is linear from $\mathcal{E}^{(\mu)}$ into $\mathcal{E}^{(\mu)}$ and hence continuous in $C^0(\bar{B}_T)$. The result is proved by applying Schauder's fixed-point theorem to $M^\dagger : \mathcal{E}^{(\mu)} \to \mathcal{E}^{(\mu)}$, which implies a fixed point $M^\dagger\mu^* = \mu^*$. □

For this μ^*, define a map $M : C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}_T^z) \to C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}_T^z)$ by $\omega' = M\omega$ with

$$\omega'(\boldsymbol{x}_H, z, t) = \int_{\bar{\Omega}_T^z} \mathrm{d}\boldsymbol{x}'_H \Gamma_U(\boldsymbol{x}_H, \boldsymbol{x}'_H, z, t, 0)\tilde{\omega}^I(\boldsymbol{x}'_H) + \int_0^t \mathrm{d}\tau \int_{\bar{\Omega}_T^z} \mathrm{d}\boldsymbol{x}'_H \Gamma_U(\boldsymbol{x}_H, \boldsymbol{x}'_H, z, t, \tau) S^{(\omega)}(\boldsymbol{x}'_H, z, \tau), \tag{5.14}$$

where $\tilde{\omega}^I$ and $S^{(\omega)}$ are given by (2.10*a*) (with $\beta = 0$) and where Γ_U is the heat potential for the linear parabolic equation

$$\omega'_t + \boldsymbol{u}(\omega)\cdot\nabla\omega' - K\Delta\omega' = S^{(\omega)}(\mu^*) \qquad \text{in} \quad \Omega_T^z. \tag{5.15}$$

Lemma 5.3. *If the conditions of lemma 5.1 are satisfied then M maps $\mathcal{E}^{(\omega)}$ into $\mathcal{E}^{(\omega)}$ provided the following hold:*

$$L^\omega - K^I \geqslant ((c_2+1)a + c_2 qT)c_2 \frac{B_1 + B_2 T L^\omega}{1 - c_2 L^\omega} + c_2(L^\omega)^2 + c_2 K^S + c_2 K^I, \tag{5.16}$$

where $c_2 = c_2(T, U, K)$ is as in lemma 5.1. Further, for any given K^I and K^S there exist values of T and L^ω that satisfy (5.16).

Proof. If $\mu \in C^0(\bar{B}_T)$ and A, Q satisfy (5.1*b*) then $\int_{\bar{B}_T} Q\mu \in C^{\nu,\frac{\nu}{2}}(\bar{B}_T)$ with $\|\int_{\bar{B}_T} Q\mu\|_{\nu;\bar{B}_T} \leqslant qT\|\mu\|_{\nu;\bar{B}_T}$ and $\int_{\bar{B}} A\mu \in C^{2+\nu}(\bar{B})$ with $|\int_{\bar{B}} A\mu|_{2+\nu;\bar{B}} \leqslant a|\mu|_{2+\nu;\bar{B}}$. Following the proof of lemma 5.1, $\omega' \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}_T^z)$ with

$$\|\omega'\|_{2+\nu;\bar{\Omega}_T^z} \leqslant c_2(qT\|\mu^*\|_{\nu;\bar{B}_T} + \|S_0\|_{\nu;\bar{\Omega}_T^z}) + (c_2+1)(a|\mu^*|_{2+\nu;\bar{B}} + |\omega^I|_{2+\nu;\bar{\Omega}}) \tag{5.17}$$

$$\leqslant ((c_2+1)a + c_2 qT)L^\mu + c_2 K^S + (c_2+1)K^I, \tag{5.18}$$

and so $\|\omega'\|_{2+\nu;\bar{\Omega}_T^z} \leqslant L^\omega_H$ provided $L^\omega_H \geqslant ((c_2+1)a + c_2 qT)L^\mu + c_2 K^S + (c_2+1)K^I$. Also,

$$\delta\omega'_t + \boldsymbol{u}|_{z_2}\cdot\nabla\delta\omega' - K\Delta\delta\omega' = \delta S^{(\omega)} - \delta\boldsymbol{u}\cdot\nabla\omega'|_{z_1} \qquad \text{in} \quad \Omega_T^{z_2}. \tag{5.19}$$

with $\delta\omega' = \delta\tilde{\omega}^I$ at $t = 0$, which gives

$$|\omega'|_{\nu;[0,h]}, |\nabla\omega'|_{\nu;[0,h]}, |\Delta\omega'|_{\nu;[0,h]} \leqslant c_2(qTL^\mu + K^S + (L^\omega)^2) + (c_2+1)(aL^\mu + K^I), \tag{5.20}$$

and so $|\omega'|_{\nu;[0,h]}, |\nabla\omega'|_{\nu;[0,h]} \leqslant L^{\omega}_{(z)}$, provided $L^{\omega}_{(z)} \geqslant ((c_2+1)a + c_2 qT)L^{\mu} + (L^{\omega})^2) + c_2 K^S + (c_2+1)K^I$. Combining the above gives $\omega' \in \mathcal{E}^{(\omega)}$ provided (5.16) holds. Finally, the right-hand side of (5.16) is zero for $T = 0$ and is positive and continuously increasing in T for $T > 0$. Thus there exists a $T > 0$ such that $c_2 L^{\omega} < 1$, and an L^{ω} with $L^{\omega} > K^I$ that satisfies (5.16). □

Theorem 5.1. *Under the assumptions (5.1a)–(5.1c), and with* $\theta \equiv 0$, *there exists a* $T > 0$ *satisfying* $c_2(T)K^I < 1$ *such that there exists a strong (classical) solution* $\omega^* \in \mathcal{E}^{(\omega)}(T)$ *of (2.8a)–(2.8d) and (2.9a)–(2.9d), where* c_2 *has the form* $c_2 = \alpha_1 T^{\frac{\nu}{2}} + \alpha_2 T^{\frac{1+\nu}{2}} + \alpha_3 T$.

Proof. As in the proof of lemma 5.2, $\mathcal{E}^{(\omega)}$ is a nonempty, convex, and compact subset of $C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T) \cap C^{\nu}([0,h])$. Also, by arguments similar to those used in lemma 4.7, M is continuous in $C^0(\bar{B}_T)$ (see also [4]). Again applying Schauder's fixed point theorem to $M : \mathcal{E}^{(\omega)} \to \mathcal{E}^{(\omega)}$ implies a fixed point $M\omega^* = \omega^*$. The fixed point is a strong solution of (2.8*a*)–(2.8*d*) and (2.9*a*)–(2.9*d*) by the construction of $\mathcal{E}^{(\omega)}$. □

5.2. *Extension to* $\beta \neq 0$, $\theta \neq 0$

First consider the case $\beta \neq 0$, $\theta \equiv 0$. The following modifications must be made to the analysis of section 2. In lemma 5.1 the bound on $\|S^{(\mu)}\|_{\nu;\bar{B}_T}$ becomes $\|S^{(\mu)}\|_{\nu;\bar{B}_T} \leqslant L^{\mu}(L^{\omega}+\beta) + B_1 + B_2 T L^{\omega}$, so that the conditions (5.9)–(5.10) become

$$c_2(L^{\omega}+\beta) < 1, \tag{5.21}$$

$$L^{\mu} \geqslant c_2 \frac{B_1 + B_2 T L^{\omega}}{1 - c_2(L^{\omega}+\beta)}. \tag{5.22}$$

Also, in lemma 5.3, the term βv must be retained in $S^{(\omega)}$, which modifies the bounds on $\|\omega'\|_{2+\nu;\bar{\Omega}^z_T}$, $|\omega'|_{\nu;[0,h]}$, $|\nabla\omega'|_{\nu;[0,h]}$, $|\Delta\omega'|_{\nu;[0,h]}$ to give the condition

$$L^{\omega} - K^I \geqslant ((c_2+1)a + c_2 qT)c_2 \frac{B_1 + B_2 T L^{\omega}}{1 - c_2(L^{\omega}+\beta)} + c_2(L^{\omega}+\beta)L^{\omega} + c_2 K^S + c_2 K^I, \tag{5.23}$$

in place of (5.16). The remainder of the proof carries over with obvious minor modifications.

Now consider the case $\theta \neq 0$, and for simplicity, and without loss of generality, assume that $\theta \neq 0$ only on Ω^0 and that $\theta \equiv 0$ on Ω^h. The Schauder estimate (5.4) for $\|\boldsymbol{u}\|_{1+\nu;\bar{B}_T}$ becomes [4]

$$\|\boldsymbol{u}\|_{1+\nu;\bar{B}_T} \leqslant c(\|\omega\|_{\nu;\bar{B}_T} + \|\theta\|_{1+\nu;\bar{\Omega}^0_T}). \tag{5.24}$$

To apply the Schauder fixed-point theorem, it is necessary to construct the following additional function spaces:

$$\mathcal{E}^{(\lambda)} = \{\lambda \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T) : \|\lambda\|_{2+\nu;\bar{\Omega}^0_T} \leqslant L^{\lambda}\}, \tag{5.25}$$

$$\mathcal{E}^{(\theta)} = \{\theta \in C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^z_T) : \|\theta\|_{2+\nu;\bar{\Omega}^0_T} \leqslant L^{\theta}\}, \tag{5.26}$$

and to define maps $N, N^{\dagger} : C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^0_T) \to C^{2+\nu,\frac{2+\nu}{2}}(\bar{\Omega}^0_T)$ by $\lambda' = N^{\dagger}\lambda$, $\theta' = N\theta$ similarly as for M and $M^{\dagger}$. The procedure is then similar to that in section 5.1 (and cf also section 4). In addition to (5.1*b*) and (5.1*c*) the following assumptions are needed:

$$\hat{Q} \in C^{\nu,\frac{\nu}{2}}(\bar{\Omega}^z_T) \times C^{\nu,\frac{\nu}{2}}(\bar{\Omega}^z_T), \qquad \hat{A} \in C^{2+\nu}(\bar{\Omega}) \times C^{2+\nu}(\bar{\Omega}) \tag{5.27a}$$

$$\theta^I \in C^{2+\nu}(\bar{\Omega}^0), \qquad H_0 \in C^{\nu,\frac{\nu}{2}}(\bar{\Omega}^0_T). \tag{5.27b}$$

First, conditions on L^{μ} and L^{λ} are found that guarantee $M^{\dagger} : \mathcal{E}^{(\mu)} \to \mathcal{E}^{(\mu)}$ and $N^{\dagger} : \mathcal{E}^{(\lambda)} \to \mathcal{E}^{(\lambda)}$ simultaneously, noting that now $\|S^{(\mu)}\|_{\nu;\bar{B}_T}$ is bounded by $c(L^{\mu}(L^{\omega}+\beta)+L^{\lambda}L^{\theta})+B_1+B_2TL^{\omega}$ and that additionally $\|H^{(\lambda)}\|_{\nu;\bar{\Omega}_T^0}$ is bounded by the same expression (cf section 4). Second, conditions on L^{ω} and L^{θ} are found that guarantee $M : \mathcal{E}^{(\omega)} \to \mathcal{E}^{(\omega)}$ and $N : \mathcal{E}^{(\theta)} \to \mathcal{E}^{(\theta)}$ simultaneously, again noting the relevant additions to the bounds on the $\|S^{(\omega)}\|_{\nu;\bar{B}_T}$ and $\|H^{(\theta)}\|_{\nu;\bar{\Omega}_T^0}$. Defining $\hat{K}^I = |\theta^I|_{2+\nu;\bar{\Omega}^0}$ and $\hat{K}^S = \|H_0\|_{\nu;\bar{\Omega}_T^0}$, the above can be summarized in the following:

Theorem 5.2. *Under the assumptions (5.1a)–(5.1c) and (5.27a)–(5.27b), there exists a* $T > 0$ *satisfying* $c_3(T)(K^I + \hat{K}^I + \beta)L < 1$ *such that there exists a strong (classical) solution* $\omega^* \in \mathcal{E}^{(\omega)}(T)$, $\theta^* \in \mathcal{E}^{(\theta)}(T)$ *of (2.8a)–(2.8d) and (2.9a)–(2.9d), where* c_3 *has the form* $c_3 = \alpha_1 T^{\frac{\nu}{2}} + \alpha_2 T^{\frac{1+\nu}{2}} + \alpha_3 T$.

6. Remarks

Theorems 4.1 and 5.2 establish the existence of a strong solution to the EL equations for the generalized inverse of the nondissipative and dissipative quasi-geostrophic equations. Existence is guaranteed in the time interval $[0, T^*]$, where $T^* < T$ decreases as the norms of the initial distributions of vorticity and temperature increase. Since T^* does not appear in any of the bounds used in establishing the proof, the problem may be recast on the interval $[0, T^*]$. Note that, even if the linear Schauder estimates of (5.4) and (5.24) are improved to logarithmic estimates of the form given in [2, equation (18)], the restriction of the time interval would remain. In the present case the main restriction arises from the coupled nonlinearity in the adjoint equations and not from the estimates of the norms of the particle paths (in the nondissipative case) or heat potentials (in the dissipative case). Of course, although the condition on T^* guarantees existence of a solution to the EL equations, such a solution is not necessarily a global minimizer of the penalty functional $\mathcal{J}$ defined in (2.6). Because $\mathcal{J}$ is nonconvex, multiple extrema may exist. This property is reflected in the fact that there is no uniqueness result for the EL equations.

It is tempting to compare the time intervals over which the existence proofs are valid between the dissipative and nondissipative cases, for example by calculating the maximal T for which (4.37) and (5.16) can be satisfied. Unfortunately such direct comparison provides no insight because of the dependence of the maximal T on K in the dissipative case, namely $T^* \sim O(K)$, which arises from the dependence of c_2 on K in (5.11) [4]. Thus in the limit of $K \to 0$, theorem 5.1 ceases to be valid at all. The reason for the discrepancy is that the dissipative equations are a higher-order system, and thus the strong solutions must have an extra order of differentiability than those of the nondissipative equations. In [5] it was speculated that in the limit of vanishing K the solutions to the dissipative equations remain at least weak solutions to the nondissipative equations.

As in [7], various extensions of the present result are possible. In particular, the advection equations for potential temperature on the upper or lower boundaries can be changed to allow for both topography and for radiation of energy out of the domain. In each case additional terms appear in both the forward and adjoint equations (2.8*c*) and (2.9*c*) for the variables θ and λ. The proof remains valid in both cases, with minor modifications: in the nondissipative case with variable topography, for example, the function specifying the topography must be $C^{2+\nu}(\bar{\Omega})$.

The existence results established here provide some justification for the numerical approximation of solutions by, say, iterative proceedures (e.g. [8, 13]). However, as illustrated in [13], different iterative proceedures approximating the same problem can provide solutions with very different convergence properties. Thus individual existence/convergence results for individual iterative schemes are still required.

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